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A Cardinality Induced Disaggregated Formulation of the Generalized Assignment Problem and Its Facets

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Abstract

We present a new disaggregated formulation of the Generalized Assignment Problem (GAP), consisting of O(mn2) variables and constraints, where m denotes the number of agents and n the number of jobs. In contrast, the traditional formulation consists of O(mn) variables and constraints. The disaggregated formulation is stronger than the traditional formulation; the linear programming relaxation of the disaggregated formulation provides tighter lower bounds. Furthermore, this new formulation provides additional opportunities for generalizations of the well-known Cover and (1, k)-Configuration inequalities that are not present in the traditional formulation. Under certain restrictive conditions, both inequalities are shown to be facets of the polytope defined by feasible solutions of GAP. We introduce two classes of inequalities involving multiple agents that are specific to this formulation. One class of inequalities is called the Bar-and-Handle (1, $\hat{p}_k$) Inequality, which under certain restrictive condition is a facet of the polytope defined by feasible solutions of GAP. Finally, we introduce another important class of inequality called the 2-Agent Cardinality Matching Inequality involving exactly two agents. Given the un-capacitated version of GAP in which each agent can process all jobs, we first show this inequality to be facets of the polytope defined by the associated bipartite graph. We then show how this inequality can be easily lifted to become a facet of the polytope defined by feasible solutions of GAP. Finally, we show that when m = 2, this inequality, along with trivial facets completely describe the polytope associated with the un-capacitated version of GAP.

Keywords: Integer Programming, Generalized Assignment, Valid Inequalities, Integer Polytope.

1. Introduction

The Generalized Assignment Problem (GAP) is a well-known problem in integer programming with numerous direct applications. More importantly, it appears as a sub-problem in many models for applications ranging from job scheduling and routing to facility location (Savelsberg 1997). Cattrysse and Van Wassenhove (1992) discuss these GAP applications and algorithms in detail.

The problem is: Let $M = \{1,\ldots, m\}$ denote a set of agents and $N = \{1,\ldots, n\}$ denote a set of jobs that need to be assigned to the agents in M. The number of units of a resource required for agent i to complete job j is a_{ij}, while the associated cost is c_{ij}. The capacity of each agent i is b_i. Without loss of generality, we assume that $a_{ij} \le b_i$ for each $i \in M$ and $j \in N$. The GAP is a decision problem that determines the minimum cost assignment of jobs in N to agents in M so that the total resources required of each agent in M does not exceed its capacity. Let $x_{ij} = 1$ if job j is assigned to agent i, 0 otherwise. The integer programming formulation of GAP is

(**P***s*) $\quad$ *Minimize* $\quad f(x) = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}$

$$s.t. \quad \sum_{j=1}^{n} a_{ij}x_{ij} \le b_i, \quad \forall i \in M \tag{1}$$

$$\sum_{i=1}^{m} x_{ij} = 1, \qquad \forall j \in N \tag{2}$$

$$x_{ij} \in \{0,1\} \qquad \forall\, i \in M,\, j \in N \tag{3}$$

In (**Ps**), (1) enforces the capacity restriction for each agent $i \in M$, while (2) specifies that every job $j \in N$ is assigned to exactly one agent in M. However, an agent may be assigned multiple jobs. Using the terminology in Gottlieb and Rao (1990a, 1990b), we refer to constraints (1) as *knapsack constraints*, and constraints (2) as *specially ordered sets* (*SOS*) *constraints*. The GAP is known to be NP-Hard and there has been considerable interest in developing algorithms to solve large instances of this problem in reasonable time.

In this paper, we present a new *disaggregated* formulation of the GAP. While this new formulation is larger than the one in (**Ps**), both in terms of the number of variables and the number of constraints, the following four reasons motivate us to examine this formulation. First, the disaggregated formulation is stronger than (**Ps**). Every feasible solution to the LP relaxation of the disaggregated formulation is feasible to the LP relaxation of (**Ps**), but not vice-versa. Since the LP relaxation of the disaggregated formulation will provide tighter bounds, any enumeration based exact procedure for GAP can benefit from the reformulation. Second, generalizations of the cover and (1, k)–configuration inequalities exist for the disaggregated formulation that are at least as strong as and far more ubiquitous than their counterparts in (**Ps**). Third, there exists classes of strong, valid inequalities involving multiple agents that are specific to the disaggregated formulation and that have no direct parallel in (**Ps**); we present two of these. Finally, even though the disaggregated formulation has more variables and constraints, the disaggregation can be done dynamically such that variables and constraints are added incrementally. Such a strategy is particularly beneficial when the Euclidean distance between the optimal solution to (**Ps**) and the optimal solution to its LP relaxation is relatively small. The success of most cutting plane methods relies on this observation and we describe such an approach in Section 6.

The rest of the paper is organized as follows. Section 2 summarizes the literature on the GAP. Section 3 presents the disaggregated formulation and shows a) that it is stronger than the standard formulation (**Ps**) and b) that well-known cover inequalities and the (1, k)-configuration inequalities for the GAP can be generalized in the disaggregated formulation by taking advantage of its structure. In section 4, we introduce the *Bar-and-Handle* (1, $\hat{p}_k$) inequality, which is unique to the disaggregated formulation and is described on an appropriate sub-graph of the bipartite graph defined for the GAP. We show that under certain restrictive conditions, this inequality is a facet of the polytope defined by the feasible solutions of GAP. In section 5, we introduce the 2-*Agent Cardinality Matching* Inequality, which again has no parallel in the original formulation. In this section, we first show that this inequality is a facet of the polytope defined by the feasible solutions to the un-capacitated version of GAP. We then show how this inequality can be easily lifted to become a facet of the polytope defined by feasible solutions of GAP. Finally, we show that for the special case of GAP consisting of just two agents, the 2-Agent Cardinality Matching inequality, along with trivial facets completely describe the polytope defined by the feasible solutions of the un-capacitated version of GAP. Section 6 concludes with implications for future research.

2. Literature Review

Algorithms to solve the GAP (**Ps**) typically involve procedures embedded in a branch-and-bound type enumeration scheme. One solution approach is to generate lower bounds by dualizing the SOS constraints (1) and solve the resulting set of knapsack problems (for example see, Ross and Soland (1975); Fisher et al. (1986); Guignard and Rosenwein (1989); Karabakal et al. (1992)). Of course, the knapsack problems themselves are NP-Hard and would require a branch-and-bound or dynamic programming procedure to solve, often employing appropriate heuristics to generate an upper bound. This basic approach is embedded in a branch-and-bound procedure to solve (**Ps**). The differences in the methodologies are largely due to the different approaches used to solve the Lagrangian dual problem. Another approach to solving the GAP is based on progressively adding columns and valid inequalities to the formulation. Savelsbergh (1997) proposed a branch-and-price algorithm for solving the GAP that involves a Dantzig-Wolf column generation scheme. This column generation scheme is embedded in a branch-and-bound procedure.

Cattryse et al. (1998) use a branch-and-cut procedure by progressively adding lifted cover inequalities to the formulation. While Cattryse et al. (1998) used LP relaxation within a branch-and-bound procedure, Nauss (2003) used Lagrangian relaxation by dualizing (1) and the cover inequalities. Avella et al. (2008) describes an approach that is similar to Cattryse et al. (1998), where facets of knapsack polytopes generated by (2) are added progressively that render the current LP solution infeasible.

Finally, Gottlieb and Rao (1990a, 1990b) provide significant insights on the convex hull of feasible solutions to (1), (2) and (3). In particular, unlike the lifted cover inequalities or the facets of single knapsacks used by Avella et al. (2008), Gottlieb and Rao (1990a, 1990b) identify classes of strong inequalities that span more than one agent. In this paper, we extend the work of Gottlieb and Rao (1990a, 1990b) by identifying valid inequalities that span more than one agent in the disaggregated formulation.

3. Disaggregated Formulation of GAP

For each agent $i \in M$, let K_i be the maximum number of jobs that agent i can handle in any feasible assignment associated with (**Ps**). We disaggregate the model **(Ps)** by separating each agent into K_i agent-cardinality combinations. More precisely, the following binary variables are now defined:

$$y_{ik} = \begin{cases} 1, \ \textit{if } k \textit{ jobs are assigned to agent } i, \\ 0, \ \textit{otherwise} \end{cases},$$

$$z_{ijk} = \begin{cases} 1, \ \textit{if job } j \textit{ is assigned to agent } i \textit{ with a cardinality of } k, \\ 0, \ \textit{otherwise}. \end{cases}.$$

The disaggregated formulation is:

(Pd) $$\textit{Minimize} \quad f(z,y) = \sum_{i \in M} \sum_{j \in N} \sum_{k=1}^{K_i} c_{ij} z_{ijk}$$

$$\textit{subject to}: \quad \sum_{j \in N} a_{ij} z_{ijk} \le b_i y_{ik}, \qquad \forall i, k = 1,..,K_i \tag{4}$$

$$\sum_{i \in M} \sum_{k=1}^{K_i} z_{ijk} = 1 \tag{5}$$

$$\sum_{j \in N} z_{ijk} = k y_{ik} \quad \forall i \in M, k = 1,..,K_i \tag{6}$$

$$\sum_{k=1}^{K_i} y_{ik} \le 1 \qquad \forall i \in M \tag{7}$$

$$z_{ijk} \le y_{ik} \qquad \forall i \in M, j \in N, k = 1,..,K_i \tag{8}$$

$$z_{ijk}, y_{ik} \in \{0,1\} \tag{9}$$

In (**Pd**), constraint set (4) represents the *knapsack constraints* defined for each agent-cardinality combination. In addition, these constraints ensure that if an agent with a given cardinality is not used then the capacity becomes zero. Constraint set (5) represents the *SOS constraints* for each job. Constraint set (6), called *cardinality constraints*, ensures that if an agent with a given cardinality is used, then the number jobs assigned to it matches its cardinality. Constraint set (7) ensures that for each agent, at most one cardinality type is used. Finally, constraint set (8), called *variable upper bound* (*VUB*) *constraints*, ensures that if an agent-cardinality combination is not used, then no job can be assigned to it.

3.1 Comparing (Ps) and (Pd)

The number of variables and constraints in (**Ps**) are both O(mn), while in (**Pd**) both are O(mn^2) since $K_i \leq n$ for all $i \in M$. We now show that (**Ps**) and (**Pd**) are equivalent. Thus, given a feasible solution to one, a feasible solution to the other can be constructed with the same objective function value as follows.

Let $x^+ \in P(s) = \{x \in R^{mn} \mid x \text{ satisfies } (1)-(3)\}$ and for each $i \in M$, $J(i) = \{j \in N \mid x_{ij}^+ = 1\}$. By definition, (i) $J(i_1) \cap J(i_2) = \emptyset$ for all $i_1, i_2 \in M$, $i_1 \neq i_2$, (ii) $\bigcup_{i \in M} J(i) = N$, and (iii) $\sum_{j \in J(i)} a_{ij} \leq b_i$ for all $i \in M$. An equivalent solution $(z^+, y^+) \in P(d) = \{(z, y) \in R^{mn^2} \mid (z, y) \text{ satisfies } (4)-(9)\}$ can be constructed as follows. For each $i \in M$, if $J(i) \neq \emptyset$, then for $k(i) = |J(i)|$, $y_{ik(i)} = 1$ and $z_{ijk(i)} = 1$ for all $j \in J(i)$, $y_{ik} = z_{ijk} = 0$ for all $k \neq k(i)$. Clearly, $f(z^+, y^+) = f(x^+)$. Conversely, for every $(z^+, y^+) \in P(d)$, a $x^+ \in P(s)$ can be constructed with $f(x^+) = f(z^+, y^+)$ as follows:

$$x_{ij}^+ = \sum_{k=1}^{K_i} z_{ijk+}, \text{ for each } i \in M, j \in N. \tag{10}$$

We now show that the LP relaxation of (**Pd**) provides a tighter bound than the LP relaxation of (**Ps**). Consider the following polytopes:

$LP(s) = \{x \in R^{mn} \mid x \text{ satisfies } (1)-(2), x \geq 0\}$,

$LP(d) = \{(z, y) \in R^p \mid (z, y) \text{ satisfies } (4)-(8), z \geq 0, y \geq 0\}$, $p = \sum_{i=1}^{m} K_i (n+1)$ and

$LP_a(s) = \{x \in R^{mn} \mid x \text{ satisfies } (10) \text{ for each } (z, y) \in LP(d), x \geq 0\}$.

Proposition 1. Let $\upsilon(LP(s)) = Min\ \{f(x) \mid x \in LP(s)\}$ and $\upsilon(LP(d)) = Min\ \{f(z,y) \mid (z,y) \in LP(d)\}$. Then $\upsilon(LP(s)) \leq \upsilon(LP(d))$.

Proof: For every $(z, y)\in LP(d)$, there exists an $x\in LP(s)$ such that $f(z, y) = f(x)$. By counterexample, we show that the reverse is not true.

For each $(z, y)\in LP(d)$, a solution $x^+\in LP_a(s)$ is obtained by aggregating z as described in (10) and aggregating y to obtain, $y_i^+ = \sum_{k=1}^{K_i} y_{ik}$ for each $i\in M$. By aggregating the constraints in (4), (5), (7) and (8) over k, it can be seen that (x^+,y^+) satisfies the constraints:

$$\sum_{j=1}^{n} a_{ij}x_{ij} \le b_i y_i, \qquad \textit{for all } i \in M$$

$$\sum_{i=1}^{m} x_{ij} = 1, \qquad \textit{for all } j \in N$$

$$x_{ij} \le y_i \qquad \textit{for all } i \in M, j \in N$$

$$x_{ij} \ge 0, 0 \le y_i \le 1, \qquad \textit{for all } i \in M, j \in N.$$

Further, since the costs c_{ij} do not vary with k, $f(x^+, y^+) = f(z, y)$. Since $y_i\le 1$ for all $i\in M$, $x^+\in LP(s)$ with $f(x^+) = f(x^+, y^+) = f(z, y)$. Thus, the above argument shows that $LP_a(s) \subseteq LP(s)$.

The following counter-example shows that in fact $LP_a(s) \subset LP(s)$. Consider a $\hat{x} \in LP(s)$ having the following characteristics:

i) Agents i_1, $i_2\in M$, with associated sets $J(i_1)\subset N$, $J(i_2)\subset N$ and $J(i_1)\cap J(i_2) = \varnothing$, such that $\sum_{j\in J(i_1)} a_{i_1 j} < b_{i_1}$ and $\sum_{j\in J(i_2)} a_{i_2 j} < b_{i_2}$.

ii) A job $j_1\in N\backslash\{J(i_1)\cup J(i_2)\}$ such that $\sum_{j\in J(i_1)} a_{i_1 j} + a_{i_1 j_1} > b_{i_1}$.

iii)(a) $\hat{x}_{i_1 j} = 1$ for all $j\in J(i_1)$ and $\hat{x}_{i_2 j} = 1$ for all $j\in J(i_2)$, (b) $\hat{x}_{i_1 j_1} = \Delta = (b_{i_1} - \sum_{j\in J(i_1)} a_{i_1 j}) / a_{i_1 j_1}$ and (c) $\hat{x}_{i_2 j_1} = 1 - \Delta$, with $\sum_{j\in J(i_2)} a_{i_2 j} + (1-\Delta) a_{i_2 j_1} \le b_{i_2}$.

By contradiction, suppose that $\hat{x} \in LP_a(s)$. Then there exists a $(\hat{z}, \hat{y})\in LP(d)$ such that $\hat{x}_{ij} = \sum_{k=1}^{K_i} \hat{z}_{ijk}$ for all $i\in M$, $j\in N$. Let $k' = |J(i_1)|$. The number of $\hat{x}_{i_1 .}$ variables that are positive is $k'+1$. Therefore, due to enforcement of constraints (6) and (8), $\hat{y}_{i_1 k} = 0$ for $k = k'+2,.., K_{i_1}$. While aggregating constraints in (6) over k, the right-hand-side obtained for agent i_1 is $\sum_{k=1}^{k'+1} k\hat{y}_{i_1k}$. From iii) it is equal to $(k'+\Delta)>k'$. Due to (7) it follows that $\hat{y}_{ik'+1}>0$. Also, $\hat{x}$ satisfies the knapsack constraint (2) associated with agent i_1 as an equality. Therefore, it can be written as $\sum_{j\in J(i_1)} a_{i_1j}(\sum_{k=1}^{k'+1}\hat{z}_{i_1jk}) + a_{i_1j_1}(\sum_{k=1}^{k'+1}\hat{z}_{i_1j_1k}) = b_{i_1}(\sum_{k=1}^{k'+1}\hat{y}_{i_1k})$.

Due to the above, every knapsack constraint (4) associated with agent i_1 and cardinality $k = 1,\ldots, k'+1$, is also satisfied as an equality. However, for $k = k'+1$, since $\sum_{j\in J(i_1)} a_{i_1j} + a_{i_1j_1} > b_{i_1}$, due to (8), it is not possible to satisfy both the knapsack constraint (4) and the cardinality constraint (6). Thus, $\hat{x} \notin LP_a(s)$, and $LP_a(s) \subset LP(s)$. □

3.2 Cardinality-Constrained Cover Inequality for (Pd)

We now present generalizations of the Cover inequalities for (**Pd**). Cover inequalities are a well-known set of inequalities derived from (1), which we generalize for (**Pd**). Let the convex hull of the 0-1 vertices of (**Ps**) and (**Pd**) be,

$H(s) = Conv\{x \in R^{mn} \mid x \text{ satisfies } (1)-(3)\}$ and

$$H(d) = Conv\{(z,y) \in R^{p} \mid (z,y) \text{ satisfies } (4)-(9)\}, p = \sum_{i=1}^{m} K_i(n+1)\} . \quad (11)$$

Using notation similar to that in Gottlieb and Rao (1990), suppose that associated with agent i, there exists a subset $N_i \subseteq N$ of jobs with $|N_i| = n_i$, such that

(i) for all $R_i \subset N_i$, $|R_i| = r_i$, $\sum_{j\in R_i} a_{ij} \le b_i$,

(ii) for all $R_{i+1} \subseteq N_i$, $|R_{i+1}| = r_i + 1$, $\sum_{j\in R_{i+1}} a_{ij} > b_i$.

Then, the following (n_i, r_i)-*cover* is a valid inequality for $H(s)$:

$$\sum_{j\in N_i} x_{ij} \le r_i . \quad (12)$$

For any (n_i, r_i)-*cover* inequality (11), it follows that the inequality,

$$\sum_{j \in N_i} z_{ijk} \le r_i y_{ik} \ , \tag{13}$$

one for each $r_i < k \le K_i$, is valid for $H(d)$.

To simplify subsequent exposition, we introduce the following optimization problem in generic terms and refer to it several times later in the paper.

Definition 1. Consider a set $D \subseteq N$ of jobs with each job $j \in D$ requiring a_j units and an integer k such that $|D| \ge k \ge 0$. Define $V^*(D, k) = Min\{\sum_{j \in D} a_j x_j \mid \sum_{j \in D} x_j = k, x \in B^{|D|}\}$.

This problem is easy to solve as it involves selecting the k smallest jobs in D. In (**Pd**), consider an agent i with a cardinality of $k \ge 2$. Suppose that there exists a set $N_i \subseteq N$, which for some $\hat{r}_{ik} < Min\{n_i, k\}$ satisfies the conditions

(a) for some $\hat{R}_{ik} \subset N_i$, $|\hat{R}_{ik}| = \hat{r}_{ik}$, $\sum_{j \in \hat{R}_{ik}} a_{ij} + V^*(N - \hat{R}_{ik}, k - \hat{r}_{ik}) \le b_i$, but (14)

(b) for all $\hat{R}_{ik+1} \subseteq N_i$, $|\hat{R}_{ik+1}| = \hat{r}_{ik} + 1$, $\sum_{j \in \hat{R}_{ik+1}} a_{ij} + V^*(N - \hat{R}_{ik+1}, k - \hat{r}_{ik} - 1) > b_i$. (15)

Then associated with each inequality (13) is a (n_i, k, $\hat{r}_{ik}$)-*cover* inequality of the form

$$\sum_{j \in N_i} z_{ijk} \le \hat{r}_{ik} y_{ik} \ , \tag{16}$$

that is valid for $H(d)$. Since $a_{ij} \le b_i$ for all $j \in N$, for the same set N_i, several distinct (n_i, k, $\hat{r}_{ik}$)-*cover* inequalities, one for each $k \ge 2$ can be derived. Since $V^*(N - \hat{R}_{ik+1}, k - \hat{r}_{ik} - 1) \ge 0$, it follows that $\hat{r}_{ik} \le r_i$ and (16) dominates (13). The following property shows that the bounds obtained from adding (n_i, k, $\hat{r}_{ik}$)-*cover* inequalities to (**Pd**) are indeed tighter than bounds obtained from adding the (n_i, r_i)-*cover* inequalities to (**Ps**).

Proposition 2 Let $LP^*(d) = \{(z,y) \in R^p \mid (z,y)\ \textit{satisfies}\ (4)-(8)\ \textit{and}\ (16)\ \textit{for every}\ i \in M, N_i \subseteq N\ \textit{and}\ k\}$ where $p = \sum_{i=1}^{m} K_i(n+1)$, and $LP^*(s) = \{x \in R^{mn} \mid x\ \textit{satisfies}\ (10)\ \textit{for each}\ (z,y) \in LP^*(d)\}$. Then, $LP^*(s) \subseteq LP(s) \cap \{x \in R^{mn} \mid x\ \textit{satisfies}\ (11)\ \textit{for every}\ i \in M, N_i \subseteq N, 0 \le x \le 1\}$.

Proof: Let $LP_s(d) = \{(z, y) \in R^p \mid (z, y)$ satisfies (7), (8) and (16) for every $i \in M$, $N_i \subseteq N$ and $k\}$ and $LP_s(s) = \{x \in R^{mn} \mid x\ \textit{satisfies}\ (10)\ \textit{for each}\ (z, y) \in LP_s(d)\}$. By definition, $LP^*(d) = LP(d) \cap LP_s(d)$ and therefore $LP^*(s) = LP_a(s) \cap LP_s(s)$. Aggregating constraints in (16) over k gives

$$\sum_{j \in N_i} x_{ij} \le \sum_{k=1}^{K_i} \hat{r}_{ik} y_{ik} ,$$

while aggregating constraints in (13) over k gives

$$\sum_{j \in N_i} x_{ij} \le \sum_{k=1}^{K_i} r_i y_{ik} .$$

Due to (7) and that $\hat{r}_{ik} \le r_i$, it follows that

$$\sum_{k=1}^{K_i} \hat{r}_{ik} y_{ik} \le \sum_{k=1}^{K_i} r_i y_{ik} \le r_i .$$

Therefore, $LP_s(s) \subseteq \{x \in R^{mn} \mid x\ \textit{satisfies}\ (12)\ \textit{for every}\ i \in M, N_i \subseteq N, 0 \le x \le 1\}$. From Proposition 2.1, $LP_a(s) \subseteq LP(s)$. Thus, $LP^*(s) \subseteq LP(s) \cap \{x \in R^{mn} \mid x\ \textit{satisfies}\ (12)\ \textit{for every}\ i \in M, N_i \subseteq N, 0 \le x \le 1\}$. □

The following example illustrates Proposition 2.2.

Example 1 Given agent i with capacity $b_i = 40$, the requirements a_{ij} on i in sorted order is {10, 10, 10, 10, 9, 9, 5, 5, 5, 3, 3, 3, 3, 3, 3}. For $N_i = \{1, 2, \ldots, 6\}$, the (6, 4)-*cover* inequality is $\sum_{j \in N_i} x_{ij} \le 4$

. Given that, $V^*(N-\hat{R}_{ik},3)=9$, $V^*(N-\hat{R}_{ik},4)=12$, $V^*(N-\hat{R}_{ik},6)=18$ and $V^*N-\hat{R}_k,7)=23$, the set of (n_i, k, $\hat{r}_{ik}$)-*cover* inequalities that can be derived for N_i are:

i) $\sum_{j\in N_i} z_{ij5} \le 3y_{i5}$, *ii*) $\sum_{j\in N_i} z_{ij6} \le 3y_{i6}$, *iii*) $\sum_{j\in N_i} z_{ij7} \le 3y_{i7}$, *iv*) $\sum_{j\in N_i} z_{ij8} \le 2y_{i8}$ and *v*) $\sum_{j\in N_i} z_{ij9} \le y_{i9}$.

Consider the partial LP solution (z, y): $y_{i8}=1, z_{i10,8}=z_{i11,8}=z_{i12,8}=z_{i13,8}=z_{i14,8}=1$, $z_{i4,8}=z_{i5,8}=z_{i6,8}=z_{i7,8}=0.75$, and $y_{ik}=z_{ijk}=0$ for all $k\neq 8$. This solution satisfies constraints (4), (6), (7) and (8) associated with agent i, but violates the (n_i, k, $\hat{r}_{ik}$)-*cover* inequality *iv*) listed above. From (z, y), a solution $x\in LP_a(s)$ can be obtained by aggregating as described in (10). This solution satisfies the (n_i, r_i)-*cover* inequality $\sum_{j\in N_i} x_{ij} \le 4$. □

As seen in Example 1 above, given a set N_i, several (n_i, k, $\hat{r}_{ik}$)-*cover* inequalities can be derived, one for each k. In a (n_i, k, $\hat{r}_{ik}$)-*cover* inequality, it is possible for $\hat{r}_{ik}$ = 0. For instance, in Example 1 above, suppose that $N_i = \{j\in N\colon a_{ij} \ge 13\}$, then for k = 9, $\hat{r}_{i9}$ = 0. The resulting (n_i, k, $\hat{r}_{ik}$)-*cover* amounts to a simple preprocessing step of setting $z_{ijk}=0$ for all $j\in N_i$. In general, this preprocessing step can be operationalized as follows.

Preprocessing Step:

For each $k \ge 2$, and $i\in M$, $j\in N$, determine $V^*(N-j,k-1)$. Set z_{ijk}=0 if $a_{ij}+V^*(N-j,k-1)>b_i$.

Let $N_{ik}\subseteq N$ be the set of z variables remaining after the preprocessing step for each i-k combination with $|N_{ik}| = n_{ik}$. The polytope

$$H(d)^{\le} = Conv\{(z,y)\in R^p \mid (z,y) \text{ satisfies } (4),(5^{\le}),(6)-(9)\},\ p=\sum_{i=1}^{m}\sum_{k=1}^{K_i} n_{ik}\}, \qquad (17)$$

where ($5^{\le}$) represents constraints (5) in less-than-or-equal-to form. Due to (6), *Dim* $\{H(d)^{\le}\} = p$.

Without loss of generality, assume the indices of N to be ordered such that $a_{ij} \geq a_{ij+1}$ for j = 1,.., n-1. Using terminology similar to that as in Balas (1975), suppose that for some $S_i \subset N_{ik}$, with $|S_i|$ = s_i, if $\sum_{j \in S_i} a_{ij} + V^*(N_{ik} - S_i, k - s_i) > b_i$, but for all $\hat{R}_{ik} \subset S_i$, $|\hat{R}_{ik}| = \hat{r}_{ik}$ and $\sum_{j \in \hat{R}_{ik}} a_{ij} + V^*(N_{ik} - \hat{R}_{ik}, k - \hat{r}_{ik}) \leq b_i$, then the ($n_i$, k, $\hat{r}_{ik}$)-*cover* inequality obtained by setting $N_i = S_i$ and $\hat{r}_{ik} = s_i - 1$ is *minimal*. The set $E(S_i) = S_i \cup S'_i$ with $S'_i = \{j \in N_{ik} - S_i \mid a_{ij} \geq a_{ij_1}\}$, where $a_{ij_1} = Max\{a_{ij} \mid j \in S_i\}$, is called an *extension* of S_i onto N_{ik}.

Definition 2. A minimal (s_i, k, $\hat{r}_{ik}$)-*cover* is *strong*, if and only if either $E(S_i) = N_{ik}$, or for the set $S''_i = S_i - \{j_1\} \cup \{j\}$, $\sum_{j \in S''_i} a_{ij} + V^*(N_{ik} - S''_i, k - s_i) \leq b_i$ for all $j \in N_{ik} - E(S_i)$.

We now proceed to show conditions under which the (n_i, k, $\hat{r}_{ik}$)-*cover* inequality (16) is a *facet* of $H(d)^{\leq}$. An inequality $\boldsymbol{gx} \leq g_0$ is a facet of a polytope F, if it is valid and if $Dim\{x \in F: \boldsymbol{gx} = g_0\} \geq Dim\{F\} - 1$. Even though our interest is in identifying facets of $H(d)$, as shown in Gottlieb and Rao (1990a), by introducing artificial variables in (5), an equivalent formulation of (**Pd**) converts it into a packing problem in which (5) is replaced by ($5^{\leq}$). Therefore, it suffices to examine facets of $H(d)^{\leq}$.

Theorem 3 The (n_i, k, $\hat{r}_{ik}$)-*cover* inequality (16) is a facet of $H(d)^{\leq}$ if the following conditions are met:

a) for some $S_i \subset N$, $N_i = E(S_i)$ with S_i being a strong minimal cover and $|S_i| = \hat{r}_{ik} + 1$,

b) $\sum_{j \in T_i} a_{ij} + V^*(N - S''_i, k - \hat{r}_{ik}) \leq b_i$, for $S''_i = \{1\} \cup S_i - \{j_1, j_2\}$, where j_1 and j_2 are the first two indices in S_i,

c) $\sum_{j \in \{S_i \setminus j_1\}} a_{ij} + \sum_{j = n_k - k + \hat{r}_i - 1}^{n_k - 1} a_{ij} \leq b_i$, and

d) $\sum_{j=n_{K_{i'}}-K_{i'}-1}^{n_{K_{i'}}-1} a_{i'j} \le b_{i'}$, for each $i' \in M$.

Proof: It suffices to identify (p-1) linearly independent, non-zero solutions $(z,y) \in \Theta = \{H(d)^{\le} |$ (16) is satisfied as an equality$\}$, since $(\mathbf{0}, \mathbf{0}) \in H(d)^{\le}$ and satisfies (16) as an equality.

The p-1 solutions are displayed as matrix Y-Z, with each row representing a $(z,y) \in \Theta$ and each column a variable y or z is

$$Y - Z = \begin{bmatrix} P_1, 0, \ldots\ldots\ldots 0 \\ 0,\ P_2, 0, \ldots\ldots 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ 0, \ldots\ldots Q_i, \ldots\ldots 0 \\ \ldots\ldots\ldots\ldots\ldots 0 \\ 0, \ldots\ldots\ldots\ldots P_m \end{bmatrix}. \qquad (18)$$

In Y-Z, sub-matrix $P_{i'}$ represents non-zero solutions associated with $i' \ne i$, while Q_i represents non-zero solutions corresponding to i. In turn, sub-matrix $P_{i'}$ consists of $K_{i'}$ sub-matrices A_l, also having a block-angular structure as

$$P_{i'} = \begin{bmatrix} A_1, 0, \ldots\ldots\ldots 0 \\ 0,\ A_2, 0, \ldots\ldots 0 \\ \ldots\ldots\ldots\ldots\ldots 0 \\ 0, \ldots\ldots\ldots\ldots A_{K_{i'}} \end{bmatrix},$$

with A_l composed of $n_{i'l}$ rows and $n_{i'l}$+1 columns, for l = 1,.., $K_{i'}$. Sub-matrix Q_i has the same structure as $P_{i'}$, except that A_k is replaced by B_k comprising of n_{ik}-1 rows and n_{ik}+1 columns. In $A_l(B_k)$, the first column represents $y_{il}(y_{ik})$ and the remaining columns represent variables $z_{ijl}(z_{ijk})$ in increasing order of j.

In A_l, the first column has all entries being one, i.e. representing y_{il} = 1. The remaining n_{il} columns in A_l are associated with the z variables and can be partitioned as

$$K_l = \begin{bmatrix} I_1 \, C_l \\ 0 \;\; D_l \end{bmatrix},$$

where I_1 is an identity matrix of order n_{il}-l-1, C_l is of dimension (n_{il}-l-1) × (l+1), D_l is of dimension (l+1) × (l+1), and **0** a matrix of zeroes of an appropriate dimension. Each row in C_l = (c_{ij}) is identical consisting of '0' in the first two entries and '1' in the rest, i.e, c_i = (0, 0, 1,…, 1). In D_l, all diagonal entries have '0' and '1' in all other locations, i.e., D_l = (d_{ij}) with d_{ij} = 0 if i = j, 1 otherwise. In B_k, the first column is composed of all entries being one, i.e. y_{ik} = 1. The columns associated with variables z_{ijk} are partitioned as

$$\Lambda_k = \begin{bmatrix} I_2 \;\; E_1 \;\; 0 \;\; F_1 \\ 0 \;\; E_2 \;\; I_3 \;\; F_2 \\ 0 \;\; E_3 \;\; 0 \;\; F_3 \\ 0 \;\; E_4 \;\; 0 \;\; F_4 \end{bmatrix}.$$

The columns in $[E_1,..,E_4]^T$ correspond to indices $j \in S_i$, while the columns in $[F_1,..,F_4]^T$ correspond to the last (k-$\hat{r}_{ik}$+1) indices in N_{ik}. I_2 is an identity matrix of order $s'_i = |S'_i|$. Its columns belong to indices in $S'_i = \{j \in N_i - S_i\}$, where $N_i = E(S_i)$. I_3 is an identity matrix of order (n_{ik}-n_i-k+$\hat{r}_{ik}$-1) whose columns are associated with indices $j = n_i+1,…, n_{ik}-k+\hat{r}_{ik}-1$. In Λ_k, **0** represents matrices of zeroes of appropriate dimension. Each row in E_1 is of the form (0, 0, 1, …, 1), while each row in E_2 and E_4 is of the form (0, 1,…,1). E_3 is a ($\hat{r}_{ik}$+1) × ($\hat{r}_{ik}$+1) matrix with diagonal entries being zero and all other entries being one. Each row in F_1 and F_3 is identical with the first entry being zero and the rest being one. In F_2 the first two entries are zero and the rest are ones. Finally, F_4 = (ϕ_{ij}) is a (k-$\hat{r}_{ik}$) × (k-$\hat{r}_{ik}$+1) matrix with ϕ_{ij} = 0 if j = i+1, 1 otherwise.

It is clear from the structure of Y-Z that each solution listed has exactly one y_{il} = 1 and the rest of the y variables set to equal zero. Since every row in K_l has exactly l entries of one and y_{il} = 1, solutions in A_l satisfy ($5^{\leq}$), (6) and (8), respectively. Due to the preprocessing step,

$a_{ij} + V^*(N-j, l-1) \le b_i$ for $1 \le j \le (n_{il}-l-1)$ and for each $2 \le l \le K_i$. These solutions correspond to rows in $[I_1\ C_l]$ which satisfy the knapsack constraints (4). Due to d), $\sum_{j=n_{il}-l-1}^{n_{il}-1} a_{ij} \le b_i$ for $2 \le l \le K_i$-1. Therefore, the solutions that correspond to $[\mathbf{0}\ D_l]$ also satisfy (4). Finally, for solutions in A_l, y_{ik} = 0 and z_{ijk} = 0 for all $j \in N_{ik}$, Therefore, (16) is satisfied as an equality. In Λ_k, since $N_i = E(S_i)$, the indices in N_i correspond to columns in $[I_2\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}]^T$ and $[E_1,.., E_4]^T$. Thus all solutions in B_k satisfy (14) as an equality. All solutions in B_k also satisfy ($5^{\le}$), (6) and (8). Condition b) describes the solution in the first row of $[I_2\ E_1\ \mathbf{0}\ F_1]$ satisfying (4). Therefore, the rest of the solutions listed in $[I_2\ E_1\ \mathbf{0}\ F_1]$ also satisfy (4). The solutions listed in $[\mathbf{0}\ E_2\ I_3\ F_2]$ satisfy (4) since S_i represents a strong minimal cover. For the same reason, the solutions in rows of $[\mathbf{0}\ E_3\ \mathbf{0}\ F_3]$ also satisfy (4). Condition c) implies that the solution listed in the last row of $[\mathbf{0}\ E_4\ \mathbf{0}\ F_4]$ satisfies (4). Hence, the remaining solutions in $[\mathbf{0}\ E_4\ \mathbf{0}\ F_4]$ satisfy (4) as well. Thus, each solution in *Y-Z* belongs to Θ, which are *p*-1 in number.

Given the block diagonal structure of *Y-Z*, in order to establish the linear independence of solutions in *Y-Z*, it is suffices to show that each A_l consists of a non-singular ($n_{il} \times n_{il}$) sub-matrix and B_k consists of a non-singular $(n_{ik}-1) \times (n_{ik}-1)$ sub-matrix. The structure of K_l is such that it is non-singular if D_l is non-singular. D_l is non-singular since $D_l^{-1} = (\kappa_{ij})$ exists and is defined as: $\kappa_{ij} = 1/l - 1$, if $i = j$, $1/l$ otherwise. Considering the sub-matrix Λ_k, suppose that by removing first column of $[F_1,..,F_4]^T$ the remaining columns are labeled $[F'_1,..,F'_4]^T$. Further, by switching the columns sets $[E_1,.., E_4]^T$ and $[\mathbf{0}\ I_3\ \mathbf{0}\ \mathbf{0}]^T$, the resulting sub-matrix obtained is Λ'_k, which is non-singular if the $(k+2)\times(k+2)$ sub-matrix $\Gamma = \begin{bmatrix} E_3 & F'_3 \\ E_4 & F'_4 \end{bmatrix}$ is non-singular. The inverse of Γ exists and is defined as

$$\Gamma^{-1} = \begin{bmatrix} X_1 & H_1 \\ X_2 & -I_4 \end{bmatrix},$$

where X_1, X_2, H_1 and $-I_4$ are of the same dimension as E_3, E_4, F'_3 and F'_4, respectively. $X_1 = (\varepsilon_{ij})$, where a) for $j = 1$, $\varepsilon_{ij} = (1-k)/\hat{r}_{ik}$ if $i=1$, $(\hat{r}_{ik}-k+1)/\hat{r}_{ik}$ otherwise, b) for $j \geq 2$, $\varepsilon_{ij} = 1/\hat{r}_{ik}-1$ if $i=j$, $1/\hat{r}_{ik}$ otherwise. Each row in X_2 is of the form $(1, 0,\ldots,0)$, while each entry in H_1 is $1/\hat{r}_{ik}$. □

It is instructive to note that if a (n_i, r_i)-*cover* inequality (12) is a facet of $H(s)^{\leq}$, then for the same N_i, a $(n_i, k, \hat{r}_{ik})$-*cover* inequality (16) with $\hat{r}_{ik} = r_i$ is a facet of $H(d)^{\leq}$. This occurs only if for some k, $\sum_{j\in\hat{R}_{ik+1}} a_{ij} > b_i$ for all $\hat{R}_{ik+1} \subseteq N_i$ in (15). For instance, in Example 1, if $a_{ij} = 2$ for all $j = 7,\ldots, 14$, the (n_i, r_i)-*cover* inequality, $\sum_{j\in N_i} x_{ij} \leq 4$ with $N_i = \{1,\ldots,6\}$, is a facet of $H(s)^{\leq}$, while the $(n_i, k, \hat{r}_{ik})$-*cover* inequality $\sum_{j\in N_i} z_{ij5} \leq 4y_{i5}$ is a facet of $H(d)^{\leq}$. Note also that the two inequalities are equivalent in the sense that for any solution $x^+\in P(s)$ that satisfies the former inequality as an equality, there is a $(z^+, y^+)\in P(d)$ that satisfies the latter as an equality, with x^+ and (z^+, y^+) satisfying (10). However, when $\sum_{j\in\hat{R}_{ik+1}} a_{ij} \leq b_i$ in (15), the inequality in $H(s)^{\leq}$ that is equivalent to the $(n_i, k, \hat{r}_{ik})$-*cover* inequality is non-canonical. For instance, consider the inequality $\sum_{j\in N_i} z_{ij7} \leq 3y_7$ in Example 1, where in $\sum_{j\in\hat{R}_{ik+1}} a_{ij} \leq b_i$, and is a facet of $H(d)^{\leq}$ when $a_{ij} = 2$ for all $j = 7,\ldots,14$. The equivalent inequality in $H(s)^{\leq}$ is $\sum_{j=1}^{6} 3x_{ij} + \sum_{j=7}^{14} x_{ij} \leq 13$, which is a facet of $H(s)^{\leq}$, but non-canonical. Thus, for several facet defining $(n_i, k, \hat{r}_{ik})$-*cover* inequalities, the equivalent inequalities in $H(s)^{\leq}$ are knapsack in nature. Herein lies the value of $(n_i, k, \hat{r}_{ik})$-*cover* inequalities in that they are easier to identify than their knapsack counterparts in $H(s)^{\leq}$.

3.3 Cardinality Constrained (1, $\hat{p}_k$)-configuration Inequalities in (Pd)

The $(1, k)$-configuration inequality is a well-known inequality used to describe the polytope of 0-1 knapsack constraints and introduced by Padberg (1980). Since in our exposition, k denotes cardinality, the same inequality is referred to as (1, p)-configuration inequality. To introduce this inequality associated with agent i, consider $N_i \subset N$, with $|N_i| = n_i$ and $q \in N \backslash N_i$, such that

(i) $\sum_{j \in N_i} a_{ij} \leq b_i$, (19)

(ii) for all sets $P \subseteq N_i$, with $|P| = p$ and $2 \leq p \leq n_i$, $\sum_{j \in P \cup q} a_{ij} > b_i$, but (20)

(iii) for all sets $P(p\text{-}1) \subseteq N_i$, with $|P(p\text{-}1)| = p\text{-}1$, $\sum_{j \in P(p-1) \cup q} a_{ij} \leq b_i$. (21)

If $R_{ik}(r_i) \subseteq N_i$ denotes a set of cardinality r_i satisfying $p \leq r_i \leq n_i$, then the (1, p)-configuration inequality in (**Ps**) is,

$$(r_i - p + 1)x_{iq} + \sum_{j \in R_{ik}(r_i)} x_{ij} \leq r_i . \tag{22}$$

It follows that for each $k \geq r_i$, the equivalent (1, p)-configuration inequality

$$(r_i - p + 1)z_{iqk} + \sum_{j \in R_{ik}(r_i)} z_{ijk} \leq r_i y_{ik} , \tag{23}$$

is valid for $H(d)$. We now present a generalization of the (1, p)-configuration inequality that is stronger than (23). Suppose that for some agent-cardinality combination i-k, N_{ik} denotes the set of jobs remaining after the preprocessing step and that $N_i \subset N_{ik}$ with $k \geq n_i$ and $q \in N_{ik} \backslash N_i$. Then,

i) $\sum_{j \in N_i} a_{ij} + V^*(N_{ik} - N_i, k - n_i) \leq b_i$, (24)

ii) for every $\hat{P}_k \subset N_i$, with $|\hat{P}_k| = \hat{p}_k$, $\sum_{j \in \hat{P}_k \cup q} a_{ij} + V^*(N_{ik} - \hat{P}_k - q, k - \hat{p}_k - 1) > b_i$, (25)

iii) but for all $\hat{P}_k(\hat{p}_k - 1) \subset N_i$, $|\hat{P}_k(\hat{p}_k - 1)| = \hat{p}_k$ -1,

$$\sum_{j \in \hat{P}_k(\hat{p}_k - 1) \cup q} a_{ij} + V^*(N_{ik} - \hat{P}_k(\hat{p}_k - 1) - q, k - \hat{p}_k) \leq b_i . \tag{26}$$

For every $R_{ik}(r_i) \subseteq N_i$ of cardinality $\hat{p}_k \leq r_i \leq n_i$, with N_i satisfying (24), (25) and (26), the following (1, $\hat{p}_k$)-configuration inequality is valid for $H(d)$

$$(r_i - \hat{p}_k + 1)z_{iqk} + \sum_{j \in R_{ik}(r_i)} z_{ijk} \leq r_i y_{ik} . \tag{27}$$

The validity of (27) follows from, i) all $(z, y) \in H(d)$ with $y_{ik} = 0$ satisfying (27) due to (8), ii) all $(z, y) \in H(d)$ with $y_{ik} = 1$ and $z_{iqk} = 0$ satisfying (27), since it follows from (24) that

$\sum_{j \in R_{ik}(r_i)} a_{ij} + V^*(N_{ik} - R_{ik}(r_i), k - r_i) \le b_i$, and iii) all $(z, y) \in H(d)$ with $y_{ik} = 1$, $z_{iqk} = 1$ satisfying (27) due to (26).

By definition, $V^*(N_{ik} - \hat{P}_k - q, k - \hat{p}_k - 1) \ge 0$. Therefore, $\hat{p}_k \le p$ and (27) dominates (23). Furthermore, several distinct (1, $\hat{p}_k$)-configuration inequalities can be constructed from the same set $R_{ik}(r_i) \cup q$, one for each $k \ge r_i$ with $\hat{p}_{k_2} \le \hat{p}_{k_1}$ for $k_2 \ge k_1$. To obtain the integer $\hat{p}_k$, set $\hat{p}_k = p$. If $V^*(N_{ik} - R_{ik}(r_i) - q, k - \hat{p}_k) + V^*(R_{ik}(r_i), \hat{p}_k - 1) + a_{iq} > b_i$, then set $\hat{p}_k = \hat{p}_k$ -1, and repeat until $V^*(N_{ik} - R_{ik}(r_i) - q, k - \hat{p}_k) + V^*(R_{ik}(r_i), \hat{p}_k - 1) + a_{iq} \le b_i$.

Proposition 4 below highlights the fact that the (1, $\hat{p}_k$)-configuration inequalities provide a tighter description of $H(d)$ than the (1, p)-configuration inequalities for $H(s)$, the proof of which can be constructed along the same lines as that for Proposition 2.

Proposition 4. Let $LP_{1p}(s) = \{x \in LP(s) |\ x$ satisfies all (1, p)-configuration inequalities (20)$\}$ and $LP_{1\hat{p}}(d) = \{(z, y) \in LP(d) |\ (z, y)$ satisfies all (1, $\hat{p}_k$)-configuration inequalities (27)$\}$. Further, $LP_{1\hat{p}}(s) = \{x \in R^{mn} \mid x$ satisfies (10) for each $(z, y) \in LP_{1\hat{p}}(d)\}$. Then, $LP_{1\hat{p}}(s) \subseteq LP_{1p}(s)$.

Let $L_i(k - \hat{p}_k + 1) = \{j \mid j = n_{ik} - k + \hat{p}_k, \ldots, n_{ik}\}$, i.e, the last $(k - \hat{p}_k + 1)$ indices in N_{ik}. Given that with $L_i(k - \hat{p}_k + 1) \cap N_i = \phi$, we denote $T_{ik} = L_i(k - \hat{p}_k + 1) \cup N_i \cup q$. We now define the polytope

$H(d)^{\le}_{i-k} = Conv\{(z, y) \in R^p |\ (z, y)$ satisfies (4), ($5^{\le}$), (6)-(9), $z_{ijk}=0$, $\forall\ j \in \{N_{ik} \backslash T_{ik}\}\}$, (28)

where $p = \sum_{i'=1}^{m} \sum_{k=1}^{K_{i'}} n_{i'k}$. Thus, $Dim\{H(d)^{\le}_{i-k}\} = p_{i-k} = p - (n_{ik} - t_{ik})$, where $t_{ik} = |T_{ik}|$.

Theorem 5. The (1, $\hat{p}_k$)-configuration inequality (27) is a facet of $H(d)^{\le}_{i-k}$ if,

a) for some $\hat{P}_{k(-1)} \subset N_i$ with $|\hat{P}_{k(-1)}| = \hat{p}_k$ -1, $\sum_{j \in \hat{P}_{k(-1)} \cup q} a_{ij} + \sum_{j = n_k - k + \hat{p}_k - 1}^{n_k - 1} a_{ij} \le b_i$, and

b) $\sum_{j=n_{K_{i'}}-K_{i'}-1}^{n_{K_{i'}}-1} a_{i'j} \le b_{i'}$ for all $i' \in M$.

Proof: The proof follows reasoning very similar to that in Theorem 3. Thus, (p_{i-k}-1) linearly independent, non-zero solutions $(z,y)\in\Theta_{i\text{-}k}$ = $\{ H(d)^{\le}_{i-k} \,|\, (r_i - \hat{p}_k + 1) z_{iqk} + \sum_{j\in R_{ik}(r_i)} z_{ijk} = r_i y_{ik} \}$ are identified, which are denoted in matrix form as Y-$Z(1, \hat{p}_k)$. Matrix Y-$Z(1, \hat{p}_k)$ is identical to Y-Z in (18), except that Q_i is replaced by Q'_i. Matrix Q'_i is identical to Q_i except that B'_k replaces B_k, with the first column associated with y_{ik} and the remaining columns associated with z_{ijk} for each $j\in T_{ik}$. Thus, it suffices to show that B'_k contains a (t_{ik}-1) × (t_{ik}-1) sub-matrix Λ'_k, which is non-singular. All the columns of Λ'_k fall under z variables with indices belonging to T_{ik}, which are arranged from left to right in the order, $\{ L_i(k-\hat{p}_k+1), q, R_{ik}(r_i), N_i \backslash R_{ik}(r_i)\}$. The t_{ik}-1 solutions $(z,y)\in\Theta_{i\text{-}k}$ that make up the rows of B'_k are listed in the same order below. They are:

i) $(z, y)^{1l}$: For each l = 1,..,k- $\hat{p}_k$, y_{ik} = 1, $z_{ij_{l+1}k} = 0$, $j_{l+1}\in L_i(k-\hat{p}_k+1)$, z_{ijk} = 1 for each $j \in \{L_i(k-\hat{p}_k+1) - j_{l+1}\}$ where j_l is the l^{th} index in $L_i(k-\hat{p}_k+1)$, z_{iqk} = 1, z_{ijk} = 1 for each $j\in L_{ik}(\hat{p}_k - 1)$ where $L_{ik}(\hat{p}_k - 1)$ are the last $(\hat{p}_k - 1)$ indices in $R_{ik}(r_i)$. The rest of the z variables are set to zero. There are (k- $\hat{p}_k$) such solutions. They are feasible due to condition a).

ii) $(z, y)^2$: y_{ik} = 1, z_{ijk} = 1 for the last (k-r_i) indices in $L_i(k-\hat{p}_k+1)$, z_{ijk} = 1 for each $j\in R_{ik}(r_i)$. The rest of the z variables are set to zero. There is one such solution, which is feasible due to (24).

iii) $(z, y)^{3l}$: y_{ik} = 1, z_{ijk} = 1 for the last (k- $\hat{p}_k$) indices in $L_i(k-\hat{p}_k+1)$ and z_{iqk} = 1. For each $1 \le l \le r_i$, if $l + \hat{p}_k \le r_i$, then for $j_u \in R_{ik}(r_i)$, $z_{ij_uk} = 1$ for each $l \le u \le l + \hat{p}_k$, $z_{ij_lk} = 0$ otherwise. Else if $l + \hat{p}_k > r_i$, then $z_{ij_lk} = 1$ for each $l \le u \le r_i$ and $1 \le u \le l + \hat{p}_k - r_i$, $z_{ij_lk} = 0$ otherwise. The z values for $j\in R_{ik}(r_i)$ is illustrated in Figure 1 below. There are r_i such solutions, all of whom satisfy (26).

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Figure 1. Illustration of z values of $(z, y)^{3l}$ for $j \in R_{ik}(r_i)$ when $r_i = 4$ and $\hat{p}_k = 2$.

iv) $(z, y)^{4l}$: $y_{ik} = 1$, $z_{ijk} = 1$ for the last $(k\text{-}r_i\text{-}1)$ indices in $L_i(k - \hat{p}_k + 1)$, $z_{ijk} = 1$ for each $j \in R_{ik}(r_i)$. For each $1 \le l \le n_i\text{-}r_i$, $z_{ij_lk} = 1$ for $j_l \in \{N_i \backslash R_{ik}(r_i)\}$. The rest of the z variables are set to zero. There are $n_i\text{-}r_i$ such solutions, all of whom are feasible as they satisfy (24).

The matrix Λ'_k is obtained by removing the columns under y_{ik} and the first index in $L_i(k - \hat{p}_k + 1)$. Consequently, Λ'_k has the following structure:

$$\Lambda'_k = \begin{bmatrix} E_1 & 1 & F_1 & 0 \\ E_2 & 0 & 1 & 0 \\ E_3 & 1 & F_3 & 0 \\ E_4 & 0 & F_4 & I_4 \end{bmatrix}.$$

In Λ'_k, column $[1\ 0\ 1\ 0]^T$ is associated with variable z_{iqk} , while columns $[0\ 0\ 0\ I_4]^T$ correspond to z variables in $N_i \backslash R_{ik}(r_i)$. Square matrix E_1 is of size $(k\text{-}\ \hat{p}_k)$, consisting of zeros along the diagonal and ones elsewhere, while E_2 is a single row consisting of ones only in the last $k\text{-}r_i$ positions. Each row in E_4 consists of ones only in the last $(k\text{-}r_i\text{-}1)$ positions, while E_3 consists only of ones. Matrix F_1 consists of ones only in the last $(\hat{p}_k - 1)$ positions in each row. The structure of F_3 is as illustrated in Figure 1, and is non-singular. Matrix F_4 contains only ones in all positions, while I_4 is an identity matrix.

Using elementary row operations, a square matrix Λ''_k, is obtained from Λ'_k as follows. Observe from Figure 1 that the $(r_i\text{-}\ \hat{p}_k + 2)^{th}$ row of F_3 consists of ones only is positions corresponding to $L_{ik}(\hat{p}_k - 1)$. The $(r_i\text{-}\ \hat{p}_k + 2)^{th}$ row of $[E_3\ 1\ F_3\ 0]$ is subtracted from each row in $[E_1\ 1\ F_1\ 0]$ to obtain $[I'_1\ 0\ 0\ 0]$, where I'_1 is an identity. In Λ''_k, $[I'_1\ 0\ 0\ 0]$ replaces $[E_1\ 1\ F_1\ 0]$. Next, rows in $[E_3\ 1\ F_3$

0] are added and the result divided by ($\hat{p}_k$ -1), which is then subtracted from [E_2 0 F_2 0] to obtain, [E'_2 -r_i/($\hat{p}_k$ -1) 0 0]. This row replaces [E_2 0 F_2 0] in Λ''_k. Since Λ''_k is a lower triangular matrix, consisting of sub-matrices I'_1, -r_i/($\hat{p}_k$ -1), F_3 and I_4, along the diagonal, all of whom are non-singular, Λ''_k is also non-singular. □

It is evident from (28) that if $N_{ik}\backslash T_{ik} = \phi$, and if the conditions specified in Theorem 5 are satisfied, then (1, $\hat{p}_k$)-configuration inequality is a facet of $H(d)^{\leq}$. If $N_{ik}\backslash T_{ik} \neq \phi$, then the variables belonging to this set can be brought into the (1, $\hat{p}_k$)-configuration inequality using a sequential lifting procedure, which is well described in Hammer et al. (1976). Specifically, in a sequential lifting procedure, the variables in $N_{ik}\backslash T_{ik}$ are brought in one by one to eventually obtain an inequality of the form

$$(r_i - \hat{p}_k + 1)z_{iqk} + \sum_{j \in R_{ik}(r_i)} z_{ijk} + \sum_{j \in N_{ik} \backslash T_{ik}} \pi_{ijk} z_{ijk} \leq r_i y_{ik} \ . \tag{29}$$

Let $V_l \subseteq N_{ik}\backslash T_{ik}$ denote the index set of l variables brought in after l iterations. The lifting procedure determines $\pi_{ij_{l+1}k}$ (coefficient of the next variable in sequence) by solving the problem,

$\pi_{ij_{l+1}k} = Min \ \{ r_i y_{ik} - (r_i - \hat{p}_k + 1)z_{iqk} + \sum_{j \in R_{ik}(r_i)} z_{ijk} + \sum_{j \in V_l} \pi_{ijk} z_{ijk} \mid$ (4), ($5^{\leq}$), (6)-(9), $z_{ij_{l+1}k} = 1$, $z_{ijk} = 0$ for all $j \in \{ N_{ik}-T_{ik}-V_l-j_{l+1} \}$}.

Suppose that q = 1, i.e. the largest item in terms of a_{ij} values, and $2 \in R_{ik}(r_i)$ (the 2nd largest item). Then, regardless of the sequence, π_{ijk} = 0 or 1, for each $j \in N_{ik}\backslash T_{ik}$. Note that if either

i) $V^*(N_{ik} - \{R_{ik}(r_i) \cup V_l \cup q \cup j_{l+1}\}, k - \hat{p}_k - 1) + V^*(R_{ik}(r_i) \cup V_l, \hat{p}_k - 1) + a_{iq} \leq b_i - a_{ij_{l+1}}$, or (30)

ii) $\sum_{j \in R_{ik}(r_i)} a_{ij} + V^*(N_{ik} - \{R_{ik}(r_i) \cup V_l \cup j_{l+1}\}, k - r_i - 1) \leq b_i - a_{ij_{l+1}}$, (31)

then $\pi_{ij_{l+1}k}$ = 0, else $\pi_{ij_{l+1}k}$ = 1. In the latter case, the result is obtained by removing item 2 from $R_{ik}(r_i)$. Thus, after lifting, (29) not only maintains the structure of the (1, $\hat{p}_k$)-configuration inequality, it is also a facet of $H(d)^{\le}$. This is of computational significance, as (1, $\hat{p}_k$)-configuration inequalities that are facets of $H(d)^{\le}$ can be identified by first setting q = 1 and ensuring that $2 \in R_{ik}(r_i)$. Then the resulting (1, $\hat{p}_k$)-configuration inequality can be lifted by applying conditions (30) and (31) (which is computationally cheap) to obtain a facet of $H(d)^{\le}$.

The following example illustrates such an instance.

Example 2. Consider agent i with $b_i = 40$ and $N_{ik} = \{1,.., 15\}$. The requirements a_{ij} in sorted order are {18, 10, 10, 10, 9, 9, 5, 5, 5, 3, 3, 3, 3, 3, 3}. For $N_i = \{2,\ldots,6\}$ and $q = 1$, one possible (1, p)-configuration inequality is: $2x_{i1}+x_{i3}+x_{i4}+x_{i5}+x_{i6} \le 4$. After sequentially lifting in sorted order, it becomes $2x_{i1}+x_{i2}+x_{i3}+x_{i4}+x_{i5}+x_{i6}+x_{i7} \le 4$, which is a facet of $H(s)^{\le}$. The equivalent (1, $\hat{p}_k$)-configuration inequality with k = 4 is, $2z_{i1k}+z_{i2k}+z_{i3k}+z_{i4k}+z_{i5k}+z_{i6k}+z_{i7k} \le 4y_{ik}$, which is a facet of $H(d)^{\le}$. However, several more (1, $\hat{p}_k$)-configuration inequalities can be derived for other values of k. The inequality $2z_{i1k}+z_{i2k}+z_{i3k}+z_{i4k}+z_{i5k}+z_{i6k} \le 3y_{ik}$ is a facet of $H(d)^{\le}$, both for k=5 and k=6. For k=7, $2z_{i1k}+z_{i2k}+z_{i3k}+z_{i4k}+z_{i5k} \le 2y_{ik}$ and $2z_{i1k}+z_{i2k}+z_{i3k}+z_{i4k}+z_{i6k} \le 2y_{ik}$ are two distinct facets of $H(d)^{\le}$. Consider the partial LP solution (z, y): $y_{i6} = z_{i26} = z_{i66} = z_{i12,6} = z_{i13,6} = z_{i14,6} = 0.5$, $z_{i1,6} = 0.3$, $z_{i15,6} = 0.2$, $y_{i7} = z_{i11,7} = z_{i12,7} = z_{i13,7} = z_{i14,7} = z_{i15,7} = 0.5$, $z_{i1,7} = z_{i2,7} = z_{i6,7} = z_{i10,7} = 0.25$. While this solution satisfies (4), ($5^{\le}$), (6) and (8), the (1, $\hat{p}_k$)-configuration inequality listed for k=6 above is violated. Also, the LP solution x, obtained by aggregating (z, y) as in (10), satisfies the (1, p)-configuration inequality listed above as well. That is because the inequality equivalent to the (1, $\hat{p}_k$)-configuration inequality for k=6 in $H(s)^{\le}$ is not a (1, p)-configuration inequality, but a more complicated knapsack inequality, which is $3x_{i1} + \sum_{j=2}^{6} 2x_{ij} + \sum_{j=7}^{15} x_{ij} \le 9$. Clearly, in terms of a separation algorithm, identifying a useful (1, $\hat{p}_k$)-configuration inequality would be easier than the knapsack inequality listed above. Herein lies the value of the (1, $\hat{p}_k$)-configuration inequality.

4.0 The Bar-and-Handle (1, $\hat{p}_k$) Inequality

We now introduce a new class of inequalities that involve more than one agent. These inequalities are defined over a 'principal' agent, along with one or more 'auxiliary' agents. Specifically, existing (1, $\hat{p}_k$)-configuration inequalities on the principal agent and cover inequalities on the auxiliary agents are used to derive the inequalities. The jobs associated with the principal agent and those associated the auxiliary agents together have a 'bar-and-handle' graph representation, hence the name.

Consider a principal agent i_p with a cardinality of k. There exists a job q and a set $N_{i_p} \subset N \setminus q$, over which several (1, $\hat{p}_k$)-configuration inequalities of type (27) exists, one for each $R_{i_p k}(r_{i_p}) \subseteq N_{i_p}$. In addition, there is a non-empty set $H \subseteq N$-$\{q \cup N_{i_p}\}$ of jobs to which the (1, $\hat{p}_k$)-configuration inequality does not extend. Specifically, for each $R_{i_p k}(\hat{p}_k - 1) \subset R_{i_p k}(r_{i_p})$,

$$\sum_{j \in \hat{P}_k(\hat{p}_k - 1) \cup q} a_{i_p j} + \sum_{j \in H} a_{i_p j} + V^*(N - \hat{P}_k(\hat{p}_k - 1) - H - q, k - \hat{p}_k - h) \leq b_{i_p}, \tag{32}$$

with $|R_{i_p k}(\hat{p}_k - 1)| =$ $\hat{p}_k$ -1.

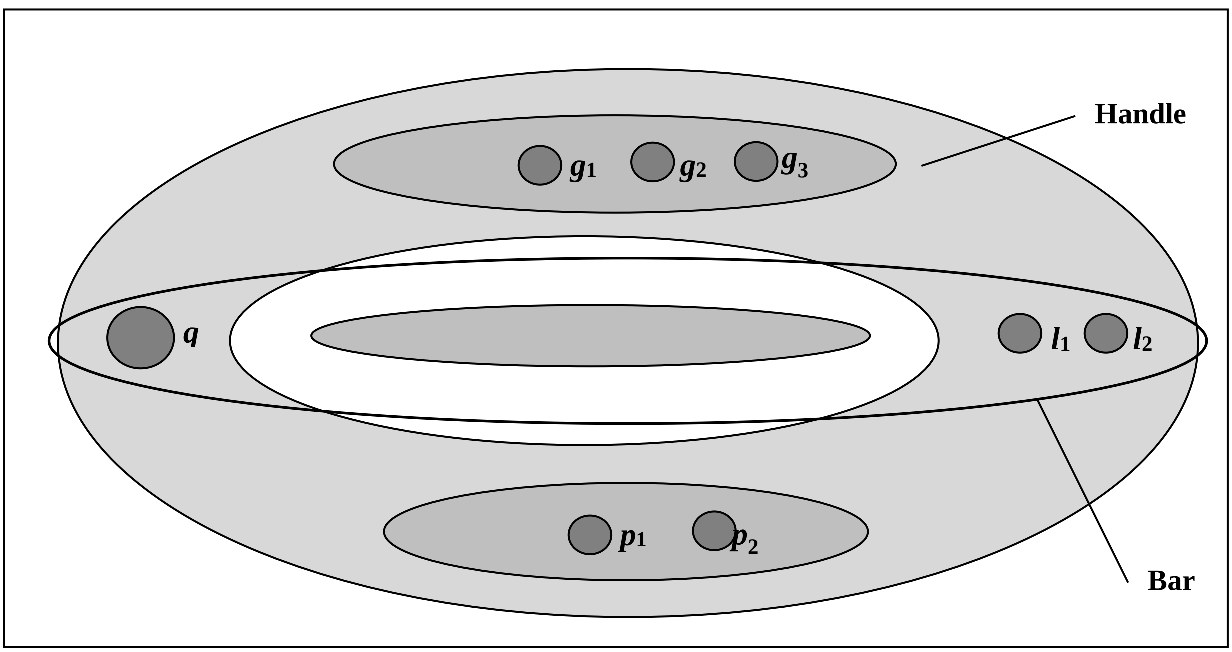

Figure 2: Illustration of Bar-and-Handle Inequalities

There also exists a set $W \subseteq \{M \setminus i_p\}$ of agents, called auxiliary agents, whose number $|W| \leq h$. For each agent $i \in W$, and with each cardinality $1 \leq k_i \leq K_i$, there either is a $(n_i(k_i), k_i, \hat{r}_{ik_i})$-cover inequality (16) defined over a set $N_i(k_i) \subseteq N\text{-}\{q \cup H\}$ with $\hat{r}_{ik_i} \leq k_i$-1, or that $n_i(k_i)= \hat{r}_{ik_i} = k_i$-1. However, if the $(n_i(k_i), k_i, \hat{r}_{ik_i})$-cover exists, it does not extend to either q or any $j \in H$. That is,

$$V^*(N_i(k_i), \hat{r}_{ik_i}) + V^*(N\text{-} N_i(k_i)\text{-} \hat{j}, k_i\text{-} \hat{r}_{ik_i} -1) + a_{i\hat{j}} \leq b_i, \tag{33}$$

for any $\hat{j} \in H$ or $\hat{j} = q$. On the other hand, if $\hat{r}_{ik_i} \leq k_i$-2, then

$$V^*(N_i(k_i), \hat{r}_{ik_i}) + V^*(N\text{-} N_i(k_i)\text{-} \hat{j} \text{-}q, k_i\text{-} \hat{r}_{ik_i} -2) + a_{i\hat{j}_1} + a_{i\hat{j}_2} > b_i \tag{34}$$

holds for every $\hat{j}_1 \in H$, $\hat{j}_2 \in \{H \cup q\}$ and $\hat{j}_1 \neq \hat{j}_2$.

The set of jobs H and q are shared by both the principal agent and the auxiliary agents. Figure 2 shows the graph of jobs associated with the principal agent and the auxiliary agents. In this figure, $W = \{i_1, i_2\}$, $H = \{l_1, l_2\}$. The set $N_{i_1}(k_1) = \{g_1, g_2, g_3\}$ and $N_{i_2}(k_2) = \{p_1, p_2\}$. It is evident from the figure that the jobs, $\{q \cup R_{i_p k}(r_{i_p}) \cup H\}$ represent the 'bar', while the jobs $\{q \cup N_i(k_i) \cup H\}$ represent the 'handle' associated with each auxiliary agent i. Thus, the bar and the handles are connected at q and H. Proposition 6 describes the Bar-and-Handle $(1, \hat{p}_k)$ inequality for (**Pd**).

Proposition 6 Let there exist:

i) A principal agent i_p, a set $N_{i_p} \subset N$ and a job $q \in \{N - N_{i_p}\}$ that satisfy conditions (24), (25) and (26), and therefore for each $R_{i_p k}(r_{i_p}) \subseteq N_{i_p}$, a $(1, \hat{p}_k)$-configuration inequality (27) can be constructed;

ii) A non-empty set $H \subseteq N\text{-}\{q \cup N_{i_p}\}$ such that (32) is satisfied with $1 \leq h \leq (k\text{-} \hat{p}_k)$ and;

iii) if $r_{i_p} < k$, then $\sum_{j \in R_{i_p k}(r_{i_p})} a_{i_p j} + a_{i_p \hat{j}} + V^*(N - R_{i_p k}(r_{i_p}), k - r_{i_p} - 1) > b_{i_p}$ for each $\hat{j} \in H$;

iv) A set $W \subseteq M \setminus i_p$ of agents, with $|W| = w \leq h$, so that for each $i \in W$ and each $1 \leq k_i \leq K_i$, there exists either a $(n_i(k_i), k_i, \hat{r}_{ik_i})$-cover inequality (16) defined over a set $N_i(k_i) \subseteq N\text{-}\{q \cup H\}$ with $\hat{r}_{ik_i} \leq k_i$-1 which satisfies (33) and (34), or that $n_i(k_i) = \hat{r}_{ik_i} = k_i$-1.

Then the inequality

$$(r_{i_p} - \hat{p}_k + 1)z_{i_pqk} + \sum_{j\in R_{i_pk}(r_{i_p})} z_{i_pjk} + \sum_{j\in H} z_{i_pjk} + \sum_{i\in W}\sum_{j\in\{q\cup N_i(k_i)\cup H\}}\sum_{k_i=1}^{K_i} z_{ijk_i} \le r_{i_p}y_{i_pk} + \sum_{i\in W}\sum_{k_i=1}^{K_i}\hat{r}_{ik_i}y_{ik_i} + h \tag{35}$$

is valid for $H(d)$.

Proof: To show that (35) is valid, consider a feasible solution $(z', y')\in P(d)$. It suffices to show that (z', y') violates (35) only if one or more conditions in i),.., iv) are not satisfied.

Define $H_1 = \{j\in H|\ z'_{i_pjk} = 1\}$ and $H_2 = \{j\in H|\ z'_{ijk_i} = 1$ for some $i\in W$, $1\le k_i\le K_i\}$, with $h_1 = |H_1|$ and $h_2 = |H_2|$. Due to (5), $h_1+h_2 \le h$. It follows that (z', y') can violate (35) only if either

$$(r_{i_p} - \hat{p}_k + 1)z'_{i_pqk} + \sum_{j\in R_{i_pk}(r_{i_p})} z'_{i_pjk} + \sum_{j\in H} z'_{i_pjk} > r_{i_p}y'_{i_pk} + h_1, \tag{36}$$

or

$$\sum_{i\in W}\sum_{j\in\{q\cup N_i(k_i)\cup H\}}\sum_{k_i=2}^{K_i} z'_{ijk_i} > \sum_{i\in W}\sum_{k_i=1}^{K_i}\hat{r}_{ik_i}y'_{ik_i} + h_2\ . \tag{37}$$

Since $\sum_{j\in H} z'_{i_pjk} = h_1$, (36) implies that either (24) or (26) of condition i), or condition ii) is not satisfied. Let $W_2 = \{i\in W|\ \sum_{k_i=1}^{K_i} y'_{ik_i} = 1\}$. One way for (37) to occur is if the $(n_i(k_i), k_i, \hat{r}_{ik_i})$-cover inequality is violated for at least one $i\in W_2$ or that $n_i(k_i) > k_i$-1. Another way is for $|W_2| = h_2$ with each $j\in H_2$ assigned to a different agent $i\in W_2$, as well as job q assigned to some agent $i\in W_2$. In either case, condition iv) is violated. A third possibility is when $|W_2| > h_2$, which occurs when all jobs in $R_{i_pk}(r_{i_p})$ are assigned to i_p, in addition to $h_1\ge 1$ jobs in H. This allows job q to be assigned to an agent $i\in W_2$. However, this would violate iii). □

Example 3 Consider the instance, $W = \{1, 2\}$, $q = 1$, $H = \{7, 8\}$, $R_{i_p4}(r_{i_p}) = \{2, 3, 4\}$, $N_{i_1}(k_1) = \{5, 6, 9\}$, $N_{i_2}(k_2) = \{6, 11\}$, $k_1 = 3$, $k_2 = 2$. The requirements on i_p are [15, 12, 12, 11, 11, 9, 7, 7, 6, 5, 5], $b_{i_p} = 41$. The requirements, $a_{1,5} = a_{1,6} = 12$, $a_{1,9} = 8$, $a_{1,1} = 3$, $a_{1,7} = a_{1,8} = 6$, $b_1 = 30$, a_{21}

= 5, $a_{2,6}$ = 9, $a_{2,11}$ = 7, $a_{2,7}$ = $a_{2,8}$ = 5, b_2 = 15. The Bar-and-Handle (1, $\hat{p}_k$) inequality obtained is

$$2z_{i_p1,4} + z_{i_p2,4} + z_{i_p3,4} + z_{i_p4,4} + z_{i_p7,4} + z_{i_p8,4} + z_{1,1,3} + z_{1,5,3} + z_{1,6,3} + z_{1,7,3} + z_{1,8,3} + z_{1,9,3} + z_{2,1,2} + z_{2,6,2}$$
$$+ z_{2,7,2} + z_{2,8,2} + z_{2,11,2} \leq 3y_{i_p4} + 2y_{13} + y_{22} + 2.$$

Consider the LP solution, y_{i_p4} =1.0, $z_{i_p1,4} = z_{i_p2,,4} = z_{i_p3,4} = z_{i_p7,4}$ = 0.5, $z_{i_p4,4}$ = 1.0, $z_{i_p10,4}$ = 1.0, y_{13} = 1.0, $z_{1,5,3} = z_{1,6,3}$ = 1.0, $z_{1,7,3} = z_{1,8,3}$ = 0.5, y_{22} = 1.0, $z_{2,11,2}$ = 0.5, $z_{2,1,2} = z_{2,8,2}$ = 0.5. While this solution satisfies (4), (5) (6), (7) and (8), the Bar-and-Handle (1, $\hat{p}_k$) inequality is violated. Note that the (1, $\hat{p}_k$)-configuration inequality, $2z_{i_p1,4} + z_{i_p,2,4} + z_{i_p,4,4} \leq 3y_{i_p4}$, as well as the ($n_i(k_i)$, k_i, $\hat{r}_{ik_i}$)-cover inequalities, $z_{1,5,3} + z_{1,6,3} + z_{1,9,3} \leq 2y_{13}$ and $z_{2,6,2} + z_{2,10,2} \leq y_{22}$ are all satisfied exactly. What this suggests is that when the (1, $\hat{p}_k$)-configuration inequality is added, its violation is removed by increasing the z values of variables in H for agent i_p. Similarly, the violation of ($n_i(k_i)$, k_i, $\hat{r}_{ik_i}$)-cover inequalities for agents $i \in W$ is removed by increasing the z values of variables in H and q. However, the resulting LP solution violates the Bar-and-Handle (1, $\hat{p}_k$), and herein lies its value.

The Bar-and-Handle (1, $\hat{p}_k$) inequality is akin to the multi-agent (1, p) Configuration inequality presented by Gottlieb and Rao (1990a, 1990b). Apart from the fact that the inequality presented in Gottlieb and Rao (1990a, 1990b) is defined over x, the principal difference is that the set of jobs shared by the principal agent and the auxiliary agents in the one presented in Gottlieb and Rao (1990a, 1990b) are $\{q \cup R_{i_pk}(r_{i_p})\}$, while in this paper the set of shared jobs are $\{q \cup H\}$. Note as well, that the Bar-and-Handle (1, $\hat{p}_k$) inequality includes the special case in which $\hat{p}_k = r_{i_p}$. This results in the 'bar' becoming a cover.

Before characterizing the strength of the Bar-and-Handle (1, $\hat{p}_k$) inequalities, we first present a general result on facets of a polytope generated by agents, $M_s \subseteq M$ and jobs $N_s \subseteq N$. Consider a sub-problem of (**Pd**) defined over a restricted set (z^r, y^r), with z^r = $\{ z_{ijk} \forall i \in M_s, j \in N_s, 1 \leq k \leq K_i \}$, and y^r = $\{ y_{ik} \forall i \in M_s, 1 \leq k_i \leq K_i \}$. The feasible set $S_1(d)^{\leq}$ = $\{(z^r, y^r) \in B^x$ | (4), ($5^{\leq}$), (6), (7), (8) and

(9)}, where $x = \sum_{i\in M_s}\sum_{j\in N_s}\sum_{k=1}^{K_i} k_i$, and the corresponding convex hull of $S_1(d)^{\leq}$ is $HS_1(d)^{\leq} = Conv\{(z,y)\in S_1(d)^{\leq}\}$.

Proposition 7 For the feasible set $S_1(d)^{\leq}$ defined by agents $M_s \subseteq M$ and jobs $N_s \subseteq N$, any nontrivial facet of $H_1(d)^{\leq}$ is an inequality of the form

$$\sum_{i\in M_s}\sum_{j\in N_s}\sum_{k=1}^{K_i}\pi_{ijk} z_{ijk} \leq \sum_{i\in M_s}\sum_{k=1}^{K_i}\pi_{ik} y_{ik} + \pi_0, \tag{38}$$

with $\pi_{ijk} \geq 0$, $\pi_{ik} \geq 0$ and $\pi_0 \geq 0$ for all $i\in M_s$, $j\in N_s$ and $1\leq k\leq K_i$.

Proof: Since $(z, y) = (0,0)\in S_1(d)^{\leq}$, it follows that for (38) to be valid, $\pi_0 \geq 0$. Suppose that $\pi_{ijk}<0$ for some $i\in M_s$, $j\in N_s$ and k. Clearly then (38) is obtained as a positive combination of inequalities that include $-z_{ijk} \leq 0$, which will later be shown as a trivial facet. If so, then (38) cannot be a facet. Hence, $\pi_{ijk\geq 0,}$ for all $i\in M_s$, $j\in N_s$ and $0\leq k\leq K_i$. Suppose that $\pi_{ik} < 0$, for some $i\in M_s$ and k. Now consider a feasible solution $(z^{r1}, y^{r1})\in S_1(d)^{\leq}$ in which $y_{ik} = 1$ and $z_{ijk} = 1$ for each $j\in N_k$ where $N_k\subset N$ and $|N_k| = k$. Clearly, there must exist another solution $(z^{r2}, y^{r2})\in S_1(d)^{\leq}$ which is identical to (z^{r1}, y^{r1}), except that $y_{ik} = 0$ and $z_{ijk} = 0$ for each $j\in N_k$. If (z^{r2}, y^{r2}) satisfies (38) exactly, then since $\pi_{ik} < 0$ and $\pi_{ijk} \geq 0$, (z^{r1}, y^{r1}) must violate (38), implying that (38) is not valid. If however, (z^{r1}, y^{r1}) satisfies (38) exactly, then $y_{ik} = 1$ for all $(z, y)\in H_1(d)^{\leq}$. Clearly, $y_{ik} \leq 1$ is not a facet, as it is dominated by (7). Hence, $\pi_{ik} \geq 0$ for all $i\in M_s$ and $0\leq k\leq K_i$. □

We now show conditions under which the Bar-and-Handle (1, $\hat{p}_k$) inequalities are facets of $H(d)^{\leq}$. Recall that $L_{i_p}(k-\hat{p}_k+1)$ consists of the last $(k-\hat{p}_k+1)$ indices in N_{i_p} in decreasing order of $a_{i_p j}$ and that $T_{i_p} = \{q\cup N_{i_p}\cup L_{i_p}(k-\hat{p}_k+1)\}$. Similarly, $L_i(k_i-\hat{r}_{ik_i}+1)$ denotes the last k_i-$\hat{r}_{ik_i}$+1 indices in N_{ik_i} for each $i\in W$ and $1 \leq k_i \leq K_i$. The complete variable set $(\boldsymbol{z^c}, \boldsymbol{y^c})$ comprises of, $\boldsymbol{z^c} = \{z_{ijk_i}\forall i\in M, j\in N_{ik_i}, k_i = 1,\ldots,K_i\}$ and $\boldsymbol{y^c} = \{y_{ik_i}\forall i\in M, k_i = 1,\ldots,K_i\}$, while the restricted variable set $(\boldsymbol{z^r}, \boldsymbol{y^r})$ comprises of $\boldsymbol{y^r} = \{y_{i_p k}, y_{ik_i}\forall i\in W, k_i = 1,\ldots,K_i\}$ and $\boldsymbol{z^r} = \{z_{i_p jk}\forall j\in[T_{i_p}\cup H],$

$z_{ijk_i} \forall i \in W$, $j \in [q \cup N_i(k_i) \cup H \cup L_i(k_i - \hat{r}_{ik_i} + 1)]$, $k_i = 1,\ldots,K_i\}$. Let $S_2(d)^{\leq} = \{(\boldsymbol{z}, \boldsymbol{y}) \in R^p |$ (4), ($5^{\leq}$), (6), (7), (8), (9), $y_{ik_i} = 0 \forall\, y \in \{\boldsymbol{y^c}\text{-}\boldsymbol{y^r}\}$, $z_{ijk_i} \forall\, z \in \{\boldsymbol{z^c}\text{-}\boldsymbol{z^r}\}\}$, where $p = \sum_{i=1}^{m} \sum_{j=1}^{n_{ik}} (k_i + 1)$. Its convex hull, $H_2(d)^{\leq} = Conv(S_2(d)^{\leq})$. Since $H_2(d)^{\leq}$ is full dimensional, $Dim\{H_2(d)^{\leq}\} = x = n_{i_p} + h + (k - \hat{p}_k + 1) + 1 + \sum_{i \in W} \sum_{k_i=1}^{K_i} (n_i(k_i) + k_i - \hat{r}_{ik_i} + 1)$.

Proposition 8 In addition to i) to iv) of Proposition 6, if the conditions:

1) for any $R_{i_p k}(r_{i_p}) \subseteq N_{i_p}$, if $a_{i_p j_1} = Max\{a_{i_p j} \mid j \in R_{i_p k}(r_{i_p})\}$, then for any $j' \in H$, $\sum_{j \in R_{i_p k}(r_{i_p}) - j_1} a_{i_p j} + a_{i_p j'} + V^*(N - N_{i_p}, k - r_{i_p}) \leq b_{i_p}$;

2) $w = h$;

3) for each $i \in W$, $1 \leq k_i \leq K_i$ and $\hat{R}_{ik_i} \subset N_i(k_i)$, a) $\sum_{j \in \hat{R}_{ik_i}} a_{ij} + a_{iq} + V^*(N_{ik_i} - N_i(k_i) - q, k - \hat{r}_{ik_i} - 1) \leq b_i$, b) $\sum_{j \in \hat{R}_{ik_i}} a_{ij} + a_{ij'} + V^*(N_{ik_i} - N_i(k_i) - j', k - \hat{r}_{ik_i} - 1) \leq b_i$ for each $j' \in H$, c) for some $j_1 \in \hat{R}_{ik_i} \subset N_i(k_i)$, $\sum_{j \in \hat{R}_{ik_i} - j_1} a_{ij} + a_{ij'} + a_{iq} + V^*(N_{ik_i} - N_i(k_i) - j' - q, k - \hat{r}_{ik_i} - 2) \leq b_i$ for each $j' \in H$; and

4) For each pair $i_1 \neq i_2$, i_1, $i_2 \in W$, there exists a $R_{i_1 k_{i_1}} \subset N_{i_1}(k_{i_1})$ and $R_{i_2 k_{i_2}} \subset N_{i_2}(k_{i_2})$ such that $R_{i_1 k_{i_1}} \cap R_{i_2 k_{i_2}} = \phi$;

are satisfied, then (35) is a facet of $H_2(d)^{\leq}$.

Proof: Consider solutions $(\boldsymbol{z^l}, \boldsymbol{y^l}) \in S_2(d)^{\leq}$, $l=1,\ldots,x$ that satisfy (35) exactly. Let $\boldsymbol{\pi^*_z z^r} \leq \boldsymbol{\pi^*_y y^r} + \pi^*_0$ be a facet inequality of $H_2(d)^{\leq}$ for which $(\boldsymbol{z^l}, \boldsymbol{y^l}) \in S_2(d)^{\leq}$, $l=1,\ldots,x$ are satisfied exactly. If so, then $\boldsymbol{\pi^*_z z^r} \leq \boldsymbol{\pi^*_y y^r} + \pi^*_0$ must be a linear multiple of (35), implying that (35) is a facet of $H_2(d)^{\leq}$. The following are a set of partial solutions, which will be used to construct $(\boldsymbol{z}, \boldsymbol{y}) \in S_2(d)^{\leq}$ that also satisfy (35) exactly. Variables not mentioned in the listing below are set to zero.

1) $(y_{i_p}, z_{i_p})^1$ = { $y_{i_pk} = 1$, $z_{i_pqk} = 1$, $z_{i_pjk} = 1 \forall j \in R_{i_pk}(\hat{p}_k - 1)$ where $R_{i_pk}(\hat{p}_k - 1) \subset R_{i_pk}(r_{i_p})$ with $|R_{i_pk}(\hat{p}_k - 1)| = \hat{p}_k - 1$, $z_{i_pjk} = 1 \forall j \in L_{i_p}(k - \hat{p}_k)$ where $L_{i_p}(k - \hat{p}_k) \subset L_{i_p}(k - \hat{p}_k + 1)$ with $|L_{i_p}(k - \hat{p}_k)| = k - \hat{p}_k$ }, which satisfies the (1, $\hat{p}_k$)-configuration inequality exactly.

2) $(y_{i_p}, z_{i_p})^2$ = {same as $(y_{i_p}, z_{i_p})^1$, except that in addition, $z_{i_pjk} = 1 \forall j \in H - j'$ and $z_{i_pj'k} = 0$, while $z_{i_pjk} = 1 \forall j \in L_{i_p}(k - \hat{p}_k - h + 1)$ }, which is feasible due to (32).

3) $(y_{i_p}, z_{i_p})^3$ = { $y_{i_pk} = 1$, $z_{i_pjk} = 1 \forall\ j \in R_{i_pk}(r_{i_p})$ and $z_{i_pjk} = 1 \forall\ j \in L_{i_p}(k - r_{i_p})$ for some $L_{i_p}(k - r_{i_p}) \subset L_{i_p}(k - \hat{p}_k + 1)$ }, which also satisfies the (1, $\hat{p}_k$)-configuration inequality exactly.

4) $(y_{i_p}, z_{i_p})^4$ = { $y_{i_pk} = 1$, for $j_1 \in R_{i_pk}(r_{i_p})$ such that $a_{i_pj_1} = Max\{a_{i_pj} \mid j \in R_{i_pk}(r_{i_p})\}$ and any $j_2 \in H$, $z_{i_pjk} = 1 \forall\ j \in R_{i_pk}(r_{i_p}) - j_1$, $z_{i_pj_2k} = 1$ and $z_{i_pjk} = 1 \forall j \in L_{i_p}(k - r_{i_p})$ for some $L_{i_p}(k - r_{i_p}) \subset L_{i_p}(k - \hat{p}_k + 1)$ }, which is feasible due to condition 1).

5) $(y_{i_p}, z_{i_p})^5$ = {same as $(y_{i_p}, z_{i_p})^3$, except that $z_{i_p\hat{j}k} = 1$ for some $\hat{j} \in N_{i_p} \setminus R_{i_pk}(r_{i_p})$ and $z_{i_pjk} = 1 \forall j \in L_{i_p}(k - r_{i_p} - 1)$ for some $L_{i_p}(k - r_{i_p} - 1) \subset L_{i_p}(k - \hat{p}_k + 1)$.

6) $(y_{ik_i}, z_{ik_i})^1$ = { $y_{ik_i} = 1$, $z_{ijk_i} = 1 \forall\ j \in \hat{R}_{ik_i}(\hat{r}_{ik_i})$ for some $\hat{R}_{ik_i}(\hat{r}_{ik_i}) \subset N_i(k_i)$ and $z_{ijk_i} = 1$ $\forall j \in L_i(k_i - \hat{r}_{ik_i})$, for some $L_i(k_i - \hat{r}_{ik_i}) \subset L_i(k_i - \hat{r}_{ik_i} + 1)$ where $i \in M$ and $1 \leq k_i \leq K_i$}, which satisfies the ($n_i(k_i)$, k_i, $\hat{r}_{ik_i}$)-cover inequality exactly.

7) $(y_{ik_i}, z_{ik_i})^2$ = {same as $(y_{ik_i}, z_{ik_i})^1$, except that $z_{ij(i)k_i} = 1$ for some $j(i) \in H$ and $z_{ijk_i} = 1$ $\forall j \in L_i(k_i - \hat{r}_{ik_i} - 1)$, for some $L_i(k_i - \hat{r}_{ik_i} - 1) \subset L_i(k_i - \hat{r}_{ik_i} + 1)$ }, satisfying condition 3b.

8) $(y_{ik_i}, z_{ik_i})^3$ = {same as $(y_{ik_i}, z_{ik_i})^1$, except that $z_{iqk} = 1$ and $z_{ijk_i} = 1$ $\forall j \in L_i(k_i - \hat{r}_{ik_i} - 1)$, for some $L_i(k_i - \hat{r}_{ik_i} - 1) \subset L_i(k_i - \hat{r}_{ik_i} + 1)$ }, satisfying condition 3a.

9) $(y_{ik_i}, z_{ik_i})^4$ = {same as $(y_{ik_i}, z_{ik_i})^2$, except that for some $j'' \in \hat{R}_{ik_i}(\hat{r}_{ik_i})$ and $1 \le k_i \le K_i$, $z_{ij''k_i} = 0$, but $z_{iqk} = 1$}, satisfying condition 3c.

We now introduce $(\boldsymbol{z}^l, \boldsymbol{y}^l) \in S_2(d)^{\le}$ that satisfy (35) exactly and progressively solve the equations $\boldsymbol{\pi}_z \boldsymbol{z}^l = \boldsymbol{\pi}_y \boldsymbol{y}^l + \pi_0$, $l = 1,..,x$, to obtain $(\boldsymbol{\pi}^*_z, \boldsymbol{\pi}^*_y, \pi^*_0)$.

I) Consider the solution $(y_{ik_i}, z_{ik_i})^2$ for each $i \in W$ such that $j(i_1) \neq j(i_2)$ if $i_1 \neq i_2$. This along with $(y_{i_p}, z_{i_p})^1$ satisfies (35) exactly. Keeping $(y_{ik_i}, z_{ik_i})^2$ fixed, r_{i_p} affinely independent solutions are obtained by varying $(y_{i_p}, z_{i_p})^1$. This is achieved by appropriately choosing that many different selections of $R_{i_pk}(\hat{p}_k - 1)$ from $R_{i_pk}(r_{i_p})$. Further, for a fixed $R_{i_pk}(\hat{p}_k - 1)$, $k - \hat{p}_k$ affinely independent solutions are obtained by selecting that many sets of $L_{i_p}(k - \hat{p}_k)$ from $L_{i_p}(k - \hat{p}_k + 1)$. Next, the solution $(y_{ik_i}, z_{ik_i})^2$ for each $i \in W$ along with $(y_{i_p}, z_{i_p})^3$, also satisfies (35) exactly. Similarly, the solution consisting of $(y_{ik_i}, z_{ik_i})^2$ for each $i \in W$ along with $(y_{i_p}, z_{i_p})^5$ also satisfy (35) exactly. Here, $(n_{i_p} - r_{i_p})$ affinely independent solutions are generated, by fixing all except choosing a different $\hat{j} \in N_{i_p} \setminus R_{i_pk}(r_{i_p})$ for each $(y_{i_p}, z_{i_p})^5$. Finally, $(y_{ik_i}, z_{ik_i})^2$ for each $i \in W$, along with $y_{i_pk} = 0$ and $z_{i_pjk} = 0 \; \forall \, j \in N$ also satisfies (35) exactly. Let $S_3(d)^{\le} = \{(z, y) \in S_2(d)^{\le} \mid (y_{ik_i}, z_{ik_i})^2$ for each $i \in W\}$ and $H_3(d)^{\le} = Conv\{S_3(d)^{\le}\}$. Given $(y_{ik_i}, z_{ik_i})^2$ for each $i \in W$, the inequality $\boldsymbol{\pi}^*_z \boldsymbol{z}^r \le \boldsymbol{\pi}^*_y \boldsymbol{y}^r + \pi^*_0$ reduces to

$$\pi_{i_pqk} z_{i_pqk} + \sum_{j \in R_{i_pk}(r_{i_p})} \pi_{i_pjk} z_{i_pjk} + \sum_{j \in L_{i_p}(k - \hat{p}_k + 1)} \pi_{i_pjk} z_{i_pjk} \le \pi_{i_p} y_{i_pk} + \sum_{i \in W} \pi_{ik_i} - \sum_{i \in W} \Big(\sum_{j \in \hat{R}_i(k_i)} \pi_{ijk_i} + \sum_{j \in L_i(k_i - \hat{r}_{ik_i} + 1)} \pi_{ijk_i} + \pi_{ij(i)} \Big) + \pi_0 ,$$

which is a facet of $H_3(d)$. We know that the $(1, \hat{p}_k)$-configuration inequality (27) defined over $\{q, N_{i_p}, L_{i_p}(k - \hat{p}_k + 1)\}$ is also a facet of $H_3(d)^{\le}$, which the $(n_{i_p} + k - \hat{p}_k + 2)$ solutions listed above satisfy exactly. It then follows that $\pi_{i_p jk} = 0$ for $j \in \{N_{i_p} - R_{i_p k}(r_{i_p}), L_{i_p}(k - \hat{p}_k + 1)\}$, $\pi_{i_p jk} = \pi_{i_p k}$ for $j \in R_{i_p k}(r_{i_p})$, $\pi_{i_p qk} = (r_{i_p} - \hat{p}_k + 1)\pi_{i_p k}$ and $\pi_{i_p} = r_{i_p}\pi_{i_p k}$.

II) Consider $(y_{i_p}, z_{i_p})^2$, $(y_{i'k_{i'}}, z_{i'k_{i'}})^2$ for some $i' \in W$ and $(y_{ik_i}, z_{ik_i})^1$ for all $i \in W \setminus i'$. Here, $z_{i_p j(i')k} = 0$, while $z_{i_p j(i')k_{i'}} = 1$. By keeping $(y_{i_p}, z_{i_p})^2$, $(y_{i'k_{i'}}, z_{i'k_{i'}})^2$ and $(y_{ik_i}, z_{ik_i})^1$ for all $i \in W - \{i', \hat{i}\}$ fixed, and varying $(y_{\hat{i}k_{\hat{i}}}, z_{\hat{i}k_{\hat{i}}})^1$, if in fact a ($n_{\hat{i}}(k_{\hat{i}}), k_{\hat{i}}, \hat{r}_{\hat{i}}$) -cover inequality exists, then $(n_{\hat{i}}(k_{\hat{i}}) + k_{\hat{i}} - \hat{r}_{\hat{i}})$ affinely independent solutions are obtained that satisfy (35) exactly. The $n_{\hat{i}}(k_{\hat{i}})$ solutions are obtained by fixing the selection of $L_{\hat{i}}(k_{\hat{i}} - \hat{r}_{\hat{i}k_{\hat{i}}})$, but making $n_{\hat{i}}(k_{\hat{i}})$ independent selections of $\hat{R}_{\hat{i}k_{\hat{i}}}(\hat{r}_{\hat{i}k_{\hat{i}}})$ from $N_{\hat{i}}(k_{\hat{i}})$. Similarly, by fixing $\hat{R}_{\hat{i}k_{\hat{i}}}(\hat{r}_{\hat{i}k_{\hat{i}}})$, $k_{\hat{i}} - \hat{r}_{\hat{i}k_{\hat{i}}}$ independent selections of $L_{\hat{i}}(k_{\hat{i}} - \hat{r}_{\hat{i}k_{\hat{i}}})$ from $L_{\hat{i}}(k_{\hat{i}} - \hat{r}_{\hat{i}k_{\hat{i}}} + 1)$ provide the remaining $k_{\hat{i}} - \hat{r}_{\hat{i}k_{\hat{i}}}$ solutions. Finally, the perturbation with $y_{\hat{i}k_{\hat{i}}} = 0$, $z_{\hat{i}jk_{\hat{i}}} = 0, \forall j \in N_{\hat{i}k_{\hat{i}}}$ also satisfies (35) exactly. Thus, after substituting out $(y_{i_p}, z_{i_p})^2$, $(y_{i'k_{i'}}, z_{i'k_{i'}})^2$ and $(y_{ik_i}, z_{ik_i})^1$ for all $i \in W - \{i', \hat{i}\}$ in $\boldsymbol{\pi}^*_z \boldsymbol{z}^r \leq \boldsymbol{\pi}^*_y \boldsymbol{y}^r + \pi^*_0$, the resulting inequality has to be a linear multiple of the ($n_{\hat{i}}(k_{\hat{i}}), k_{\hat{i}}, \hat{r}_{\hat{i}}$) -cover inequality. Thus, $\pi_{\hat{i}jk_{\hat{i}}} = 0$, $\forall j \in L_{\hat{i}}(k_{\hat{i}} - \hat{r}_{\hat{i}k_{\hat{i}}} + 1)$, $\pi_{\hat{i}jk_{\hat{i}}} = \pi_{\hat{i}}$, $j \in N_{\hat{i}}(k_{\hat{i}})$ and $\pi_{\hat{i}k_{\hat{i}}} = \hat{r}_{\hat{i}k_{\hat{i}}}\pi_{\hat{i}}$. Since the choice of $\hat{i} \in W$ is arbitrary and the above holds for all $1 \leq k_{\hat{i}} \leq K_{\hat{i}}$, $\pi_{ijk_i} = 0$, for all $j \in L_i(k_i - \hat{r}_{ik_i} + 1)$, $\pi_{ijk_i} = \pi_i$ for all $j \in N_i(k_i)$ and $\pi_{ik_i} = \hat{r}_{ik_i}\pi_i$, for each $i \in W$. Suppose that for some $i' \in W$, $1 \leq k_{i'} \leq K_{i'}$, $\hat{R}_{i'k_{i'}}(\hat{r}_{i'k_{i'}}) = N_{i'}(k_{i'})$ and $\hat{r}_{i'k_{i'}} = k_{i'}$-1 in (35). If so, then consider $(y_{i_p}, z_{i_p})^3$, $(y_{i'k_{i'}}, z_{i'k_{i'}})^4$ for some $i' \in W$ and $(y_{ik_i}, z_{ik_i})^2$ for all $i \in W \setminus i'$. Note that in $(y_{i'k_{i'}}, z_{i'k_{i'}})^4$, $z_{i'j''k_{i'}} = 0$ for some $j'' \in \hat{R}_{i'k_{i'}}(\hat{r}_{i'k_{i'}})$. By keeping $(y_{i_p}, z_{i_p})^3$ and $(y_{ik_i}, z_{ik_i})^2$ fixed for all $i \in \{W - i'\}$ and vary $(y_{i'k_{i'}}, z_{i'k_{i'}})^4$ by j'', we get $k_{i'} - 1$ solutions, all of which satisfy (35) exactly. The choice of i' being arbitrary, here as well, $\pi_{ijk_i} = \pi_i$ for all $j \in N_i(k_i)$ and $\pi_{ik_i} = \hat{r}_{ik_i}\pi_i$, for each $i \in W$.

III) Consider again $(y_{i_p}, z_{i_p})^2$, $(y_{i'k_{i'}}, z_{i'k_{i'}})^2$ for some $i' \in W$ and $(y_{ik_i}, z_{ik_i})^1$ for all $i \in W \setminus i'$, with $z_{i' j_1 k_{i'}} = 1$, i.e., $j(i') = j_1$. If this solution is perturbed by setting $z_{i' j_1 k_{i'}} = 0$ and $z_{i_p j_1 k} = 1$, then it satisfies (35) exactly as well. Comparing both solutions we get, $\pi_{i_p j_1 k} = \pi_{i' j_1 k_{i'}}$. Since the choice of j_1 and i' is arbitrary, $\pi_{i' j_1 k_{i'}} = \pi_{i_p k}$ for all $j_1 \in H$ and $i' \in W$.

IV) Now consider $(y_{i_p}, z_{i_p})^3$, $(y_{\hat{i}k_{\hat{i}}}, z_{\hat{i}k_{\hat{i}}})^3$ for some $\hat{i} \in W$, $(y_{ik_i}, z_{ik_i})^2 \forall i \in W \setminus \hat{i}$. Since $w = h$, there is a job $j' \in H$ which is unassigned. This solution is perturbed by setting $z_{\hat{i}qk_{\hat{i}}} = 0$ and $z_{\hat{i}j'k_{\hat{i}}} = 1$, which also satisfies (35) exactly. By comparing the two solutions we get $\pi_{\hat{i}qk_{\hat{i}}} = \pi_{\hat{i}j'k_{\hat{i}}}$ for $j' \in H$. Since the choice of $j' \in H$ and $\hat{i} \in W$ is arbitrary, it follows that $\pi_{\hat{i}qk_{\hat{i}}} = \pi_{\hat{i}j'k_{\hat{i}}} = \pi_{i_p k}$ for each $j' \in H$, $\hat{i} \in W$ and $1 \le k_{\hat{i}} \le K_{\hat{i}}$. Finally, from the perturbed solution we obtain $\pi_0 = h\pi_{i_p k}$.

V) Consider, $(y_{i_p}, z_{i_p})^3$, $(y_{\hat{i}k_{\hat{i}}}, z_{\hat{i}k_{\hat{i}}})^4$ for some $\hat{i} \in W$, $(y_{ik_i}, z_{ik_i})^2 \forall i \in W \setminus \hat{i}$. By comparing this solution to that in IV) we get $\pi_{\hat{i}j''k_{\hat{i}}} = \pi_{\hat{i}j'k_{\hat{i}}} = \pi_{i_p k}$ for each $j'' \in N_{\hat{i}}(k_{\hat{i}})$ and $j' \in H$. Since the choice of $\hat{i} \in W$ is arbitrary, it follows that $\pi_{ijk_i} = \pi_i = \pi_{i_p k}$ for each $i \in W$ and $j \in N_i(k_i)$.

This establishes $\boldsymbol{\pi}^*_z \boldsymbol{z}^r \le \boldsymbol{\pi}^*_y \boldsymbol{y}^r + \pi^*_0$ to be a scalar multiple of (35) and therefore (35) is a facet of $H_2(d)^{\le}$. □

We now proceed to show conditions under which the Bar-and-Handle (1, $\hat{p}_k$) inequality (35) is a facet of $H(d)^{\le}$, the largest dimensioned polytope. We now define a knapsack polytope $H_{\hat{i}}(d)^{\le}$ = $Conv\{S_{\hat{i}}(d)^{\le}\}$, where $S_{\hat{i}}(d)^{\le}$ = $\{(\mathbf{z}, \mathbf{y}) \in R^p \mid$ (4), (5$^{\le}$), (6), (7), (8), (9), $y_{ik_i} = 0$ and $z_{ijk_i} = 0$ for all $i \ne \hat{i}$, $j \in N$ and $1 \le k_i \le K_i\}$, where $p = \sum_{i=1}^{m} \sum_{j=1}^{n_{ik}} (k_i + 1)$.

Theorem 9 The Bar-and-Handle (1, $\hat{p}_k$) inequality (35) is a facet of $H(d)^{\le}$, if in addition to conditions 1), 2), 3) and 4) of Proposition 8, it also satisfies conditions:

1) the (1, $\hat{p}_k$)-configuration inequality (27) defined over $\{q, R_{i_p k}(r_{i_p})\}$ is a facet of $H_{i_p}(d)^{\le}$;

2) the ($n_i(k_i)$, k_i, $\hat{r}_{ik_i}$)-cover inequality (16) for each $i \in W$ and $1 \le k_i \le K_i$ is a facet of $H_i(d)^{\le}$;

3) for each $\hat{i} \in M - \{W, i_p\}$, a) $K_{\hat{i}} \le$ ($n_{\hat{i}K_{\hat{i}}}$ -h), b) $V^*(N_{\hat{i}K_{\hat{i}}} - H - \hat{j}, K_{\hat{i}} - 1) + a_{\hat{i}\hat{j}} \le b_{\hat{i}}$ for each $\hat{j} \in \{N_{\hat{i}K_{\hat{i}}} - H\}$ and c) $V^*(N_{\hat{i}K_{\hat{i}}} - H - q, K_{\hat{i}} - 1) + a_{\hat{i}\hat{j}} \le b_{\hat{i}}$ for each $\hat{j} \in H$;

4) if $K_{i_p} \ne k$, then a) $K_{i_p} \le (n_{i_pK_{i_p}} - h)$, b) $V^*(N_{i_pK_{i_p}} - H - \hat{j}, K_{i_p} - 1) + a_{i_p\hat{j}} \le b_{i_p}$ for each $\hat{j} \in \{N_{i_pK_{i_p}} - H\}$ and c) $V^*(N_{i_pK_{i_p}} - H - q, K_{i_p} - 1) + a_{i_p\hat{j}} \le b_{i_p}$ for each $\hat{j} \in H$.

Proof: Starting with (35), a sequential lifting procedure will lift coefficients of variables $y_{ik} \in \{\boldsymbol{y^c}\text{-}\boldsymbol{y^r}\}$ and $z_{ijk} \in \{\boldsymbol{z^c}\text{-}z^r\}$, the lifted coefficients being π_{ik} and π_{ijk}, respectively. The resulting inequality obtained will be a facet of $H(d)^{\le}$. It therefore suffices to show that the lifted coefficients of variables in $\{\boldsymbol{y^c}\text{-}\boldsymbol{y^r}\}$ and $\{\boldsymbol{z^c}\text{-}z^r\}$ are all zero.

Let U_y and U_z denote the index set of all y and z variables in $\{\boldsymbol{y^c}\text{-}\boldsymbol{y^r}\}$ and $\{\boldsymbol{z^c}\text{-}z^r\}$, respectively, while $V_y \subseteq U_y$ and $V_z \subseteq U_z$ represent those that have already been lifted. If the coefficient of $y_{ik_i}(l)$ for $l \in \{U_y\text{-}V_y\}$ is to be lifted next, then it can be determined by solving the problem

$$\begin{aligned}
&\pi_{ik_i}(l) = Min\{r_{i_p} y_{i_pk} + \sum_{i \in W}\sum_{k_i=1}^{K_i} \hat{r}_{ik_i} y_{ik_i} + h - (r_{i_p} - \hat{p}_k + 1) z_{i_pqk} - \sum_{j \in R_{i_pk}(r_{i_p})} z_{i_pjk} - \sum_{j \in H} z_{i_pjk} \\
&- \sum_{i \in W}\sum_{j \in \{q \cup N_i(k_i) \cup H\}}\sum_{k_i=1}^{K_i} z_{ijk_i} - \sum_{s \in V_y} \pi_{ik}(s) y_{ik}(s) - \sum_{s \in V_z} \pi_{ijk}(s) z_{ijk}(s) \,|\, (4),(5^{\le}),(6),(7),(8),(9), \qquad (39) \\
&y_{ik_i}(l) = 1, y_{ik}(s) = 0, \forall s \in \{U_y - V_y - l\}, z_{ijk}(s) = 0, \forall s \in \{U_z - V_z\}\}.
\end{aligned}$$

To determine $\pi_{ijk_i}(l)$, for $l \in \{U_z\text{-}V_z\}$, the problem solved is the same as in (39), except that $z_{ijk_i}(l) = 1$ and $z_{ijk}(s) = 0$, $\forall s \in \{U_z\text{-}V_z\text{-}l\}$.

Consider first the coefficients of variables $y_{\hat{i}k_{\hat{i}}}(l)$ in which $l \in \{U_y\text{-}V_y\}$ and $\hat{i} \notin \{W, i_p\}$. For the first variable selected, the solution to (39) is $(y_{i1}, z_{i1})^2$ for each $i \in W$ in which $z_{ij(i)1} = 1$ for $j(i) \in H$ with $j(i_1) \ne j(i_2)$ if $i_1 \ne i_2$, and $\hat{r}_{i1} = k_i - 1 = 0$ for each $i \in W$. Observe that with this solution, due to conditions 3a) and 3b), it is possible to feasibly assign $k_{\hat{i}} \le K_{\hat{i}}$ jobs to $\hat{i}$, from the set $\{N_{\hat{i}K_{\hat{i}}} - H\}$. Consequently, $\pi_{\hat{i}k_{\hat{i}}}(1)$=0. The coefficients of subsequent variables $y_{\hat{i}k_{\hat{i}}}(l)$ in which $l \in \{U_y\text{-}V_y\}$ and $\hat{i} \notin \{W, i_p\}$ will also be zero, since $\pi_{\hat{i}k_{\hat{i}}}(s) = 0$ for $s \in V_y$, and therefore the same solution to (39) holds good. Due to 3b), the same solution holds for variables $z_{\hat{i}\hat{j}k_{\hat{i}}}(l)$ in which $\hat{i} \notin \{W, i_p\}$ and $\hat{j} \notin H$. Therefore, the lifted coefficients of these variables are zero as well. For variables $z_{\hat{i}\hat{j}k_{\hat{i}}}(l)$ in which $\hat{i} \notin \{W, i_p\}$ but $\hat{j} \in H$, the optimal solution to (39) is $(y_{i1}, z_{i1})^2$ for each $i \in \{W \backslash i'\}$ in which $z_{ij(i)1} = 1$ for $j(i) \in H$ with $j(i_1) \neq j(i_2)$ if $i_1 \neq i_2$ and $(y_{i'1}, z_{i'1})^3$ for i' in which $z_{i'q1} = 1$. Due to condition 3c), with this solution it is possible to assign $k_{\hat{i}} \le K_{\hat{i}}$ jobs to $\hat{i}$, which includes $\hat{j}$. Therefore, $\pi_{\hat{i}\hat{j}k_{\hat{i}}}(l) = 0$ for each $l \in \{U_z\text{-}V_z\}$ in which $\hat{i} \notin \{W, i_p\}$ and $\hat{j} \in H$.

Consider next the variables $y_{i_p k_{i_p}}(l)$ in which $l \in \{U_y\text{-}V_y\}$ and $k_{i_p} \neq k$. If $k_{i_p} = K_{i_p}$, then the optimal solution to (39) is also $z_{ij(i)1} = 1$ for each $j(i) \in H$ with $j(i_1) \neq j(i_2)$ if $i_1 \neq i_2$, and $\hat{r}_{i1} = k_i - 1 = 0$ for each $i \in W$. Here, due to 4a) and 4b), K_{i_p} jobs from $\{N_{i_p K_{i_p}} - H\}$ can be feasibly assigned to i_p. Therefore, the same holds true for any $k_{i_p} < K_{i_p}$ and $\pi_{i_p k_{i_p}}(l) = 0$ for all $k_{i_p} \neq k$. The same solution also applies for variables $z_{i_p j k_{i_p}}(l)$ for each $j \in \{N_{i_p K_{i_p}} - H\}$ and therefore their lifted coefficients will be zero due to 4b). For variables $z_{i_p j k_{i_p}}(l)$ in which $j \in H$, the optimal solution to (39) is $(y_{i1}, z_{i1})^2$ for each $i \in \{W\text{-}i'\}$ in which $z_{ij(i)1} = 1$ for $j(i) \in \{H\text{-}j\}$ with $j(i_1) \neq j(i_2)$ if $i_1 \neq i_2$ and $(y_{i'1}, z_{i'1})^3$ for i' with $z_{i'q1} = 1$. Due to 4c), one can assign $k_{i_p} < K_{i_p}$ jobs from the set $\{N_{i_p K_{i_p}} - \{H \setminus j\}\}$ to i_p, which includes $j \in H$. Therefore, $\pi_{i_p j k_{i_p}}(l) = 0$ for each $j \in H$, $l \in \{U_z\text{-}V_z\}$.

For variables $z_{i_p jk}(l)$ in which $l \in \{U_z\text{-}V_z\}$ and $j \in \{N_{i_p k} - \{q, N_{i_p}, L_{i_p}(k - \hat{p}_k + 1), H\}\}$, since $R_{i_p k}(\hat{p}_k - 1) \subset R_{i_p k}(r_{i_p}) \subseteq N_{i_p}$, it follows from 1), that $V^*(R_{i_p k}(r_{i_p}), \hat{p}_k - 1) + a_{i_p q} + a_{i_p j} +$

$V^*(L_{i_p}(k-\hat{p}_k+1), k-\hat{p}_k-1) \le b_{i_p}$. Otherwise, either $j \in R_{i_p k}(r_{i_p})$, or that the (1, $\hat{p}_k$)-configuration inequality defined over {q, $R_{i_p k}(r_{i_p})$} is a not facet of $H_{i_p}(d)^{\le}$. Given the above, the optimal solution to (39) is $(y_{ik_i}, z_{ik_i})^2$ for each $i \in W$ in which $z_{ij(i)1} = 1$ for $j(i) \in H$ with $j(i_1) \neq j(i_2)$ if $i_1 \neq i_2$, and $(y_{i_p}, z_{i_p})^1$, with $z_{i_p jk}(l) = 1$. This results in $\pi_{i_p jk}(l) = 0$ for each $l \in \{U_z\text{-}V_z\}$ and $j \in \{N_{i_p k} - \{q, N_{i_p}, L_{i_p}(k - r_{i_p} + 1), H\}\}$. For variables $z_{ijk}(l)$, in which $i \in W$, $j \in \{N_{ik_i} - N_i(k_i)\}$ and $l \in \{U_z\text{-}V_z\}$, due to condition 2), $V^*(N_i(k_i), \hat{r}_{ik_i}) + a_{ij} + V^*(L_i(k_i - \hat{r}_{ik_i} + 1), k_i - \hat{r}_{ik_i} - 1) \le b_i$, for each $j \in \{N_{ik_i} - N_i(k_i)\}$. Otherwise, either $j \in N_i(k_i)$ or that the ($n_i(k_i)$, k_i, $\hat{r}_{ik_i}$)-cover inequality is not a facet of $H_i(d)^{\le}$. Thus, $\pi_{ijk_i}(l) = 0$ for each $l \in \{U_z\text{-}V_z\}$ and $j \in \{N_{ik_i} - N_i(k_i) - L_i(k_i - \hat{r}_i + 1)\}$. □

5.0 2-Agent Cardinality Matching Inequality

The 2-Agent Cardinality Matching Inequality presented in this section is derived from an un-capacitated version of (**Pd**), i.e. one without the knapsack constraints (4). The intuition behind the inequality is to account for how jobs are matched to agents. The left-hand-side of the inequality describes the revealed potential assignment of jobs to agents in terms of the *z* variables. The right-hand-side represents the total available assignment slots described in terms of *y* variables.

5.1 Construction of 2-Agent Cardinality Matching Inequality

Consider an agent pair $W = \{i_1, i_2\}$ and a set of jobs $H \subset N$ with the following specifications:

1) Agents i_1 and i_2 are identified with specific cardinalities of k_1 and k_2 respectively, such that either i) $k_1 \ge (n+1)\text{-}K_{i2}$, ii) $k_2 \ge n\text{-}K_{i1}$ but iii) $k_1+k_2 \le n$, or that k_1 and k_2 is determined by the support $0 < y_{i_1 k_1} < 1$, and $0 < y_{i_2 k_2} < 1$, but with $k_1+k_2 \le n$.
2) $H_q \subset N$, with $H_q = \{j_{q1}, j_{q2}, j_{q3}, j_{q4}\}$,
3) $K_{i1}+K_{i2} \ge n$, i.e, agents i_1 and i_2 alone can completely accommodate all jobs.

To help explain the construction of the 2-Agent Cardinality Matching inequality, consider the following equality, obtained by aggregating the cardinality constraints (6) over *i* and *k*:

$$\sum_{i\in M-W}\sum_{j\in N}\sum_{k_i=1}^{K_i} z_{ijk_i} + \sum_{i\in W}\sum_{j\in N}\sum_{k_i=1}^{K_i} z_{ijk_i} = \sum_{i\in M-W}\sum_{k_i=1}^{K_i} k_i y_{ik_i} + \sum_{i\in W}\sum_{k_i=1}^{K_i} k_i y_{ik_i} \tag{40}$$

First, a set of *z* variables are selectively removed from the left-hand-side of (40) as follows. They are: i) *z* variables that assign job j_{q1} to agent-cardinality combinations (i_1, k_{i1}) for $1 \le k_{i1} \le k_1$-1, (i_2, k_{i2}) for $k_{i2} \ge k_2$+1 or to all the agent-cardinalities in *M*-*W*, ii) *z* variables that assign job j_{q2} to the agent-cardinality combinations (i_2, k_{i2}) for $1 \le k_{i2} \le k_2$, (i_1, k_{i1}) for $k_1 \le k_{i1} \le K_{i1}$, or to all the agent-cardinalities in *M*-*W*, iii) *z* variables that assign job j_{q3} to agent-cardinality combinations (i_1, k_{i1}) for $1 \le k_{i1} \le k_1$-1, (i_2, k_{i2}) for $1 \le k_{i2} \le k_2$, or to all the agent-cardinality combinations in *M*-*W*, and iv) *z* variables that assign job j_{q4} to agent-cardinality combinations (i_1, k_{i1}) for $k_1 \le k_{i1} \le K_{i1}$-1, ($i_2$, k_{i2}) for k_2+1 $\le k_{i2} \le K_{i2}$-1, and to all agent-cardinality combinations (i, k_i), where $i \in M$-W and $1 \le k_i \le \hat{k} = n$-$k_1$-$k_2$-1.

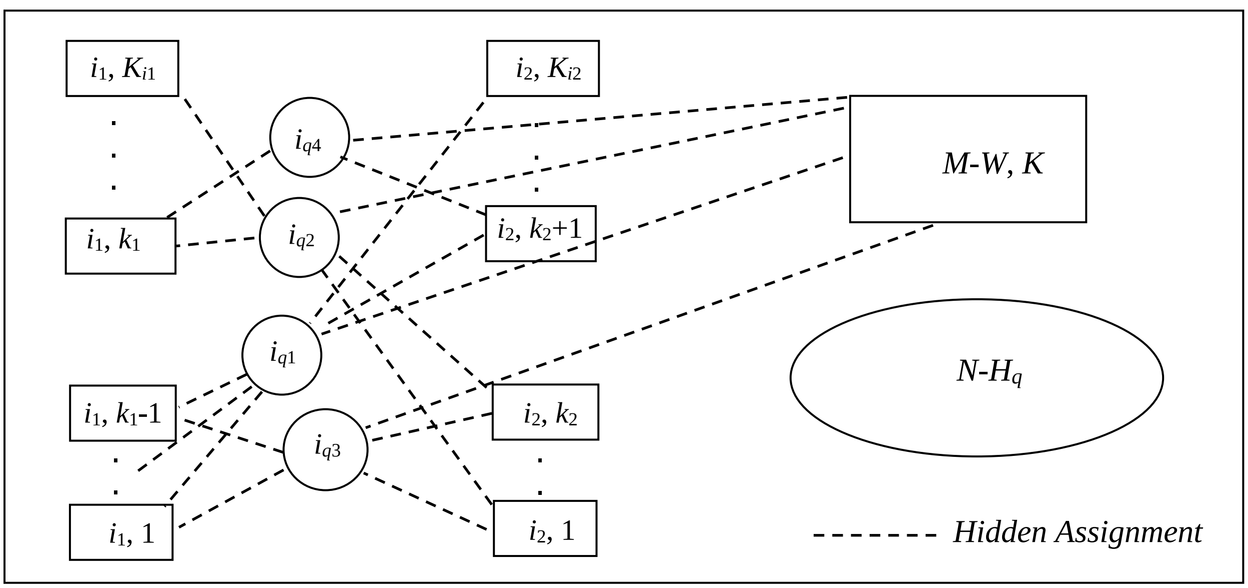

Figure 3. Illustration of hidden assignments in 2-Agent Cardinality Matching Inequality

We refer to these removed variables as ‘hidden’ assignments. The graph in Figure 3 illustrates some of these hidden assignments.

Next, the coefficients of variables $y_{i1,k_{i1}}$ and $y_{i2,k_{i2}}$, decrease by 1, i.e. becomes k_{i1}-1 and k_{i2}-1, respectively. Finally, a constant of 1 is added to the right-hand-side. The 2-Agent Cardinality Matching Inequality can be stated as:

$$\sum_{j\in N-\{j_{q1},j_{q3}\}}\sum_{k_{i1}=1}^{k_1-1} z_{i_1jk_{i1}} + \sum_{j\in N-\{j_{q2},j_{q4}\}}\sum_{k_{i1}=k_1}^{K_{i1}-1} z_{i_1jk_{i1}} + \sum_{j\in N-\{j_{q2}\}} z_{i_1jK_{i1}} + \sum_{j\in N-\{j_{q2},j_{q3}\}}\sum_{k_{i2}=1}^{k_2} z_{i_2jk_{i2}} + \sum_{j\in N-\{j_{q1},j_{q4}\}}\sum_{k_{i2}=k_2+1}^{K_{i2}-1} z_{i_2jk_{i2}} +$$
$$\sum_{j\in N-\{j_{q1}\}} z_{i_2jK_{i2}} + \sum_{i\in M-W}\Big(\sum_{j\in\{N-H_q\}}\sum_{k_i=1}^{\hat{k}} z_{ijk_i} + \sum_{j\in\{N-j_{q1}-j_{q2}-j_{q3}\}}\sum_{k_i=\hat{k}+1}^{K_i} z_{ijk_i}\Big) \le \sum_{k_{i1}=1}^{K_{i1}}(k_{i1}-1)y_{i_1k_{i1}} + \sum_{k_{i2}=1}^{K_{i2}}(k_{i2}-1)y_{i_2k_{i2}} +$$
$$\sum_{i\in M-W}\sum_{k_i=1}^{K_i} k_i y_{ik_i} + 1 \qquad (41)$$

Proposition 10 Consider, i) an agent pair $W = \{i_1, i_2\}$ associated with specific cardinalities $\{k_1, k_2\}$ such that $k_1+k_2 \le n$, ii) $k_1 \ge (n+1)-K_{i2}$, $k_2 \ge n-K_{i1}$ and $\hat{k} = n-k_1-k_2-1$, and iii) $H_q \subset N$ consists of nodes $\{j_{q1}, j_{q2}, j_{q3}, j_{q4}\}$. Then the 2-Agent Cardinality Matching Inequality (41) is valid for $H(d)$.

Proof: The argument below shows that all feasible solutions to (**Pd**) satisfy (41). The feasible solutions to (**Pd**) can be categorized into the following cases, each satisfying (41).

Case I: Consider feasible solutions in which no agent in W is used. Then, due to (8), (41) reduces to $\sum_{i\in M-W}\sum_{j\in N-H_q}\sum_{k_i=1}^{K_i} z_{ijk_i} \le \sum_{i\in M-W}\sum_{k_i=1}^{K_i} k_i y_{ik_i} + 1$, which is satisfied due to (6) and (8).

Case II: Consider feasible solutions that use agent-cardinalities (i_1, k_{i1}) and (i_2, k_{i2}), where $k_{i1} \ge k_1$ and $k_{i2} \le k_2$. Due to (5), all n jobs are assigned. Observe that due to (6), agents in M-W with appropriate cardinalities are used so as to accommodate n-k_{i1}-k_{i2} jobs. Consequently, the right-hand-side of (41) is $(k_{i1}-1)+(k_{i2}-1)+(n-k_{i1}-k_{i2})+1 = n-1$. Observe also that the assignment job j_{q2} is hidden. Therefore, the left-hand-side of (41) is at most n-1, and the constraint is satisfied.

Case III: Consider feasible solutions that use agent-cardinalities (i_1, k_{i1}) and (i_2, k_{i2}), with $k_{i1} \le k_1$-1 and $k_{i2} \ge k_2$+1. Here, as with Case II, due to (5), the left-hand-side of (41) is at most n-1, since the assignment of j_{q1} is hidden, while the right-hand-side is n-1.

Case IV: Consider feasible solutions that use agent-cardinalities (i_1, k_{i1}) and (i_2, k_{i2}), with $k_{i1} \le k_1$-1 and $k_{i2} \le k_2$. Here as well, the right-hand-side of (41) is $(k_{i1}-1)+(k_{i2}-1)+(n-k_{i1}-k_{i2})+1 = n-1$. However, since the assignment of job j_{q3} is hidden, the left-hand-side is at most $(n-1)$.

Case V: Consider feasible solutions using agent-cardinalities (i_1, k_{i1}) and (i_2, k_{i2}), with $k_{i1} \geq k_1$ and $k_{i2} \geq k_2$+1. Note that in this situation, only agent-cardinalities (i, k_i), where $i \in M$-W, $k_i \leq \hat{k}$ can be used. Further, $k_{i1}+k_{i2} \leq n$. If so, then as before, the right-hand-side of (41) is $(k_{i1}-1)+(k_{i2}-1)+(n-k_{i1}-k_{i2})+1 = n-1$. Since the assignment of j_{q4} is hidden from (i_1, k_{i1}), (i_2, k_{i2}) and all agents in M-W, the left-hand-side is at most n-1.

Case VI: Consider feasible solutions where exactly one agent-cardinality belonging to either i_1 or i_2 are used. The rest of the agent-cardinalities used belong to {M-W, K}. If agent-cardinality (i_1, k_{i1}) is used with $k_{i1} \leq k_1$, then the assignment of jobs j_{q1} and j_{q3} are hidden. If (i_1, k_{i1}) is used with $k_{i1} \geq k_1$+1, then the assignment of jobs j_{q2} and j_{q4} are hidden. Similarly, if agent-cardinality (i_2, k_{i2}) is used with $k_{i2} \leq k_2$-1, then the assignment of jobs j_{q2} and j_{q3} are hidden, while if $k_{i2} \geq k_2$, then jobs j_{q2} and j_{q3} are hidden. In all the four cases above, the left-hand-side of (41) is at most n-2, while the right-hand-side is n. □

Example 6: This example illustrates a feasible LP solution that violates (41). Let W = {i_1, i_2}, N = {1, 2, 3, 4, 5, 6, 7, 8}, H_q = {j_{q1}, j_{q2}, j_{q3}, j_{q4}} = {2, 3, 4, 5}, $k_1 = k_2 = 3$. The partial LP solution is: $y_{i_i 3} = y_{i_2 3} = y_{i_{i_2 4}} = y_{i_1 2} = 0.5$, $y_{i_c 2} = 1.0$ (here $i_c \in M$-W), $z_{i_1 1,3} = z_{i_1 2,3} == z_{i_1 4,3} = z_{i_1 1,2} = z_{i_1 3,2} = 0.5$,

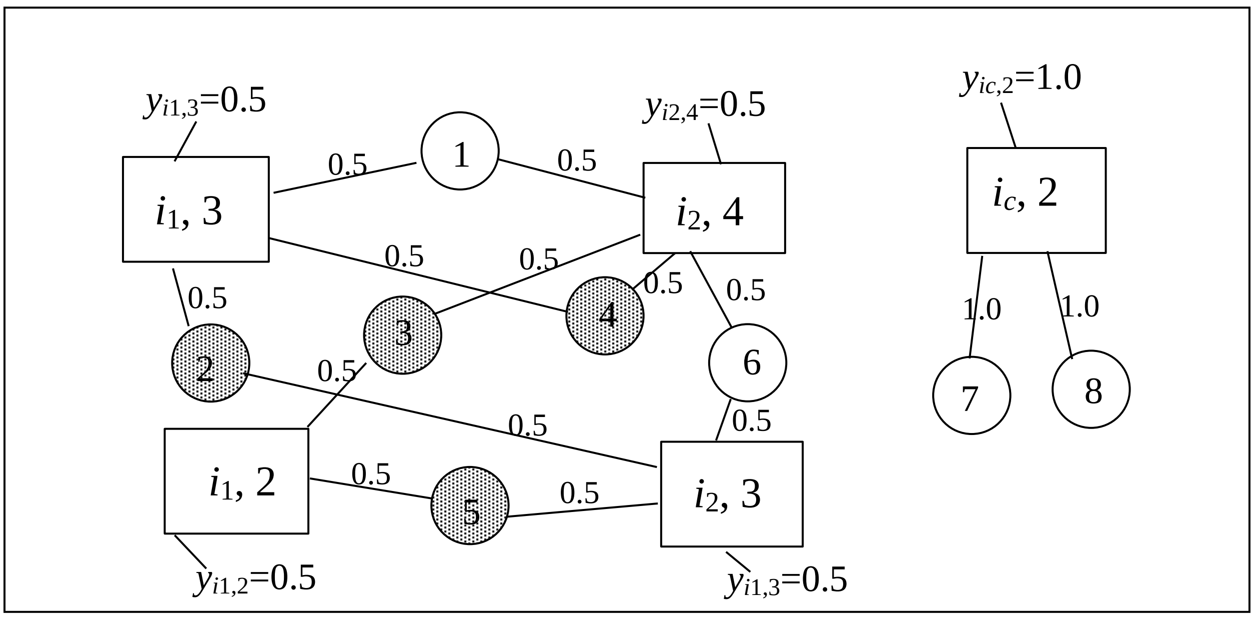

Figure 4. Illustration of Example 6.

$z_{i_2 3,4} = z_{i_2 4,4} = z_{i_2 5,4} = z_{i_2 6,4} = z_{i_2 2,3} = z_{i_2 5,3} = z_{i_2 6,3} = 0.5$, $z_{i_c 7,2} = z_{i_c 8,2} = 1$. This LP solution satisfies (5), (6), (7) and (8). The left-hand-side of (41) is 8, while the right-hand-side is 7.0. Hence, (41) is violated. This example is illustrated in Figure 4 above.

5.2 Strength of 2-Agent Cardinality Matching Inequality

We now examine the strength of the 2-Agent Cardinality Matching Inequality. Let

$$HU(d)^{\leq} = Conv\{(z,y) \in R^p \mid (z,y) \text{ satisfies } (5^{\leq}),(6)-(9)\},\ p = \sum_{i\in M}(n+1)K_i\} \tag{42}$$

Note that $HU(d)^{\leq}$ describes that polytope of an un-capacitated version of GAP, i.e. without knapsack constraints (4). We now make an important, but reasonable assumption that $K_i \leq n$-4 for each $i \in M$.

Proposition 11. *Dim* $\{HU(d)^{\leq}\} = \sum_{i\in M} nK_i$.

Proof: We identify the following integer solutions $(z,y) \in HU(d)^{\leq}$ which are linearly independent noting that (**0**, **0**) is also feasible. For each $i \in M$, $1 \leq k \leq K_i$, $y_{ik} = 1$, $z_{ijk} = 1$ for each $j \in N_k$, where $N_k \subset N$ with $|N_k| = k$, and the rest of the integer (z,y) variables set to zero. Note that it is possible to make n linearly independent selections of N_k from N as long as $k < n$. Since $K_i \leq n$-3 for each $i \in M$, the number of such linearly independent solutions is $\sum_{i\in M} nK_i$. □

Proposition 12. The inequalities, a) $x_{ijk} \geq 0$ for all $i \in M$, $j \in N$, $1 \leq k \leq K_i$, b) $z_{ijk} \leq y_{ik}$ for all $i \in M$, $j \in N$, $1 \leq k \leq K_i$, c) *SOS* constraints ($5^{\leq}$), and d) constraint (7) for each $i \in M$ are *trivial* facets of $HU(d)^{\leq}$.

Proof: For a) and b), we identify $\sum_{i\in M} nK_i$ -1 linearly independent integer solutions in $HU(d)^{\leq}$ that satisfy the respective inequalities as an equality, noting that (**0**, **0**) also satisfies them as an equality.

Consider first the inequalities, $x_{ijk} \geq 0$ for all $i \in M$, $j \in N$, $k \leq K_i$. Here, for each $i' \neq i$, as well as $i' = i$, but $k' \neq k$, we identify solutions as described in the proof for Proposition 11. For the case where $i' \neq i$, there are $\sum_{i' \in \{M-i\}} nK_{i'}$ such solutions. For the case where $i' = i$ and $k' \neq k$, there are $(K_i$-1$)n$ such solutions. Finally, for $i' = i$ and $k' = k$, we can make n-1 linearly independent selections of N_k from

N-j since $k < n$-1, giving us n-1 additional integer solutions in $HU(d)^{\leq}$. This results in a total of $\sum_{i\in M} nK_i$ -1 linearly independent integer solutions.

Consider next the inequalities $z_{ijk} \leq y_{ik}$ for all $i\in M$, $j\in N$, $1 \leq k \leq K_i$. We first set $y_{ik} = z_{ijk} = 0$. Then, for each $i' \neq i$, as well as $i' = i$, but $k' \neq k$, we identify some of those solutions described in the proof for Proposition 11. The total number of such solutions are $\sum_{i'\in\{M-i\}} nK_{i'}$ + $n(K_i$-1). We now set y_{ik} = z_{ijk} = 1, and $y_{i'k'}$ = 0, $z_{i'j'k'}$ = 0 for all $i'\neq i$, as well as for $i'=i$ but $k'\neq k$. Let $N_{k-1}\subset\{N-j\}$ with $|N_{k-1}|$ = k-1. Along with $y_{ik} = z_{ijk}$ = 1, we set $z_{ij'k}$ = 1 for each $j'\in N_{k-1}$. Clearly, there are n-1 independent selections of N_{k-1} possible from $\{N-j\}$. Thus, a total of $\sum_{i\in M} nK_i$ -1 linearly independent integer solutions are obtained.

Now consider the *SOS* constraint ($5^{\leq}$) for some $j\in N$. We identify $\sum_{i\in M} nK_i$ affinely independent solutions in $HU(d)^{\leq}$ that satisfy it an equality, which does not include (**0**, **0**). Consider first the case, k = 1. Here, y_{i1} = 1, z_{ij1} = 1, and $y_{i+1,K_{i+1}} = 1$ and $z_{i+1,j',K_{i+1}}$ = 1 for each $j'\in N_{K_{i+1}} \subset\{N-j\}$, for some $1\leq i\leq m$-1, satisfies ($5^{\leq}$) exactly. Since $K_{i+1} \leq n$-3, it is possible to obtain n-1 linearly independent selections of $N_{K_{i+1}}$ from $\{N-j\}$. In addition, the solution, y_{i1} = 1, z_{ij1} = 1 alone satisfies ($5^{\leq}$) exactly as well. This set of solutions can be recreated by varying i from 1 to m-1. For $i = m$, let y_{m1} = 1, z_{mj1} = 1, $y_{1,K_1} = 1$ and z_{1,j',K_1} = 1 for each $j'\in N_{K_{i+1}} \subset\{N-j\}$. Here again, it is possible to obtain n-1 linearly independent selections of $N_{K_{i+1}}$ from $\{N-j\}$. In addition, consider the solution y_{m1} = 1, z_{mj1} = 1 alone. Thus, from this class, a total of mn solutions are obtained. Now consider cases where $2\leq k\leq K_i$. Here, let y_{ik} = 1, z_{ijk} =1, $z_{ij'k}$ = 1 for each $j'\in N_{k-1}$, where $|N_{k-1}| = k$-1, $N_{k-1}\subset\{N-j\}$. There are n-1 linearly independent selections of N_{k-1} from $\{N-j\}$, resulting in n-1 solutions. In addition, for $i \leq m$-1, the solution $y_{i+1,1}$ = 1, $z_{i+1,j'',1}$ = 1 for $j''\in\{N-N_{k-1}-j\}$ is appended to y_{ik} = 1, z_{ijk} =1, $z_{ij'k}$ = 1 for each $j'\in N_{k-1}$, for some selection N_{k-1}. For i=m, the solution $y_{1,1}$=1, $z_{1,j'',1}$ = 1 for $j''\in N'_{n-k-1}=\{N-N_{k-1}-j\}$ is appended. Thus, by varying k from 1 to K_i, for each $i\in M$, a total of $\sum_{i\in M} nK_i$

solutions are obtained. If all the $\sum_{i\in M} nK_i$ solutions identified were placed in sorted order of indices *i*, *j* and *k*, it is easy to see that it has a block-angular structure, and hence affinely independent.

Finally, consider constraint (7) for some $i'\in M$. As with the *SOS* constraints, we identify $\sum_{i\in M} nK_i$ affinely independent solutions in $HU(d)^{\leq}$ that satisfy it an equality. For agent i', $1\leq k\leq K_{i'}$, $y_{i'k} = 1$, $z_{i'jk} = 1$ for each $j\in N_k$, where $N_k\subset N$ with $|N_k| = k$, and the rest of the integer (z,y) variables set to zero. Given that n linearly independent selections of N_k from N can be made, a total of $nK_{i'}$ affinely independent solutions are obtained. Next, for each $i\in M\text{-}i'$ and $1 \leq k_i \leq K_i$, we let $y_{i'1} = 1$, $z_{i'11} = 1$, $y_{ik_i} = 1, z_{ijk_i} = 1$ for each $j \in N_{k_i}$, where $N_{k_i} \subset\{N\text{-}1\}$ with $|N_{k_i}| = k_i$, and the rest of the integer (z,y) variables set to zero. For each k_i, one obtains n-1 affinely independent solutions. Therefore, add the solution, $y_{i'1} = 1$, $z_{i'21} = 1$, $y_{i''1} = 1$, $z_{i''11} = 1$. Thus, we obtain a total of $\sum_{i\in\{M-i'\}} nK_i$ linearly independent solutions. □

Let the complete variable set $(\boldsymbol{z^c}, \boldsymbol{y^c})$ comprises of, $\boldsymbol{z^c} = \{z_{ijk_i} \forall i \in M,\ j \in N_{ik_i}, k_i = 1,\ldots,K_i\}$ and $\boldsymbol{y^c} = \{y_{ik_i} \forall i \in M, k_i = 1,\ldots,K_i\}$. We now redefine the restricted variable set $(\boldsymbol{z^r}, \boldsymbol{y^r})\in B^x$, as consisting of all *z* and *y* variables that appear in constraint (41). Specifically, $\boldsymbol{z^r}$ is obtained by removing from $\boldsymbol{z^c}$, all hidden *z* variables, while $\boldsymbol{y^r} = \boldsymbol{y^c}$.

Proposition 12 Let $S_4(d)^{\leq} = \{(\boldsymbol{z}, \boldsymbol{y})\in R^x|\ (5^{\leq}), (6), (7), (8), (9),\ z_{ijk_i} = 0\ \forall\ z \in\{\boldsymbol{z^c}\text{-}\boldsymbol{z^r}\}\}$, with *x* of appropriate dimension. The 2-Agent Cardinality Matching Inequality (41) is a non-trivial facet of $H_4(d)^{\leq} = Conv\{(z,y) \in S_4(d)^{\leq}\}$.

Proof: Let $\boldsymbol{\pi^*_z z^r} \leq \boldsymbol{\pi^*_y y^r} + \pi_0$ be a non-trivial facet inequality of $H_4(d)^{\leq}$. Using solutions $(\boldsymbol{z^l}, \boldsymbol{y^l})\in S_4(d)^{\leq}$, that satisfy (41) exactly, we show that $\boldsymbol{\pi^*_z z^r} = \boldsymbol{\pi^*_y y^r} + \pi_0$ must be a linear multiple of (41) as an equality, implying that (41) is a facet of $H_4(d)^{\leq}$. The inequality, $\boldsymbol{\pi^*_z z^r} \leq \boldsymbol{\pi^*_y y^r} + \pi_0$ in expanded form is

$$\sum_{j\in N-\{j_{q1},j_{q3}\}}\sum_{k_{i1}=1}^{k_1-1}\pi_{i_1jk_{i1}}z_{i_1jk_{i1}}+\sum_{j\in N-\{j_{q2},j_{q4}\}}\sum_{k_{i1}=k_1}^{K_{i1}-1}\pi_{i_1jk_{i1}}z_{i_1jk_{i1}}+\sum_{j\in N-\{j_{q2}\}}\pi_{i_1jK_{i1}}z_{i_1jK_{i1}}+\sum_{j\in N-\{j_{q2},j_{q3}\}}\sum_{k_{i2}=1}^{k_2}\pi_{i_2jk_{i2}}z_{i_2jk_{i2}}+$$
$$\sum_{j\in N-\{j_{q1},j_{q4}\}}\sum_{k_{i2}=k_2+1}^{K_{i2}-1}\pi_{i_2jk_{i2}}z_{i_2jk_{i2}}+\sum_{j\in N-\{j_{q1}\}}z_{i_2jK_{i2}}+\sum_{i\in M-W}\sum_{j\in N-H_q}\sum_{k_i=1}^{K_i}\pi_{ijk_i}z_{ijk_i}\sum_{j\in\{N-j_{q1}-j_{q2}-j_{q3}\}}\sum_{k_i=\hat{k}+1}^{K_i}\pi_{ijk_i}z_{ijk_i}\le$$
$$\sum_{k_{i1}=1}^{K_{i1}}\pi_{i_1k_{i1}}y_{i_1k_{i1}}+\sum_{k_{i2}=1}^{K_{i2}}\pi_{i_2k_{i2}}y_{i_2k_{i2}}+\sum_{i\in M-W}\sum_{k_i=1}^{K_i}\pi_{ik_i}y_{ik_i}+\pi_0. \tag{43}$$

We now examine feasible solutions to $S_4(d)^{\le}$ that satisfy (41), as well as (43) exactly. All listings of feasible solutions below mention only non-zero variables.

Consider a feasible solution $(z, y)^1$: i) $y_{i_1 2}=y_{i_2 2}=1$, ii) $y_{ik_i}=1$ for a permutation of each $i\in M_w\subseteq M-W$ and cardinalities k_i chosen such that $\sum_{i\in M_W}k_i=n-4$, iii) jobs j_{q2} and j_{q4} are assigned to $(i_1, 2)$, j_{q1} and j_{q3} are assigned to $(i_2, 2)$, and iv) the remaining jobs in N-H_q are assigned to agents in M_W. Note that since the assignment of j_{q3} is hidden, (41) is satisfied exactly. This solution is perturbed by choosing another job $j\in N$-H_q that is currently assigned to agent-cardinality $(\hat{i},k_{\hat{i}})$, where $\hat{i}\in M_W$ and exchange its assignment with j_{q3}. That is, j is assigned to $(i_2, 2)$, while j_{q3} is assigned to $(\hat{i},k_{\hat{i}})$. The perturbed solution satisfies (41) and therefore (43) exactly as well. Therefore, $\pi_{i_2j2}=\pi_{\hat{i}jk_{\hat{i}}}$ for $1\le k_{\hat{i}}\le K_{\hat{i}}$. A similar result, where $\pi_{i_1j2}=\pi_{\hat{i}jk_{\hat{i}}}$ for $1\le k_{\hat{i}}\le K_{\hat{i}}$ can be obtained if in $(z, y)^1$, j_{q3} were assigned to (i_1, k_{i1}) in place of j_{q4}. Since the choice of $\hat{i}\in M_W$ and $k_{\hat{i}}$ is arbitrary, it follows that $\pi_{\hat{i}jk_{\hat{i}}}=\pi a_j$ for all $\hat{i}\in M_W$, $j\in\{N$-$H_q\}$ and $1\le k_{\hat{i}}\le K_{\hat{i}}$. Another perturbation of $(z, y)^1$ occurs by exchanging the assignment of j_{q3} and j_{q4}, wherein j_{q3} is now assigned to (i_1, k_{i1}). This perturbation shows that $\pi_{i_1j_{q4}2}=\pi_{i_2j_{q4}2}$. This result can be generalized for any $k_{i1}\le k_1$-1 and $k_{i2}\le k_2$, which results in $\pi_{i_1j_{q4}k_{i1}}=\pi_{i_2j_{q4}k_{i2}}=\pi a_j$ for all $k_{i1}\le k_1$-1 and $k_{i2}\le k_2$.

Consider a feasible solution $(z, y)^2$: i) $y_{i_1k_{i1}}=y_{i_2k_{i2}}=1$, where $k_1\le k_{i1}\le K_{i1}$-1, $1\le k_{i2}\le k_2$ and $k_{i1}+k_{i2}\le n$-1, ii) $y_{ik_i}=1$ each $i\in M_w\subseteq M-W$ and cardinalities k_i chosen such that $\sum_{i\in M_W}k_i=n-k_{i1}-k_{i2}\ge 1$, iii) k_{i1} jobs in $\{N$-j_{q2}-$j_{q4}\}$, including j_{q1} and j_{q3} are assigned to (i_1, k_{i1}), iv) remaining k_{i2} jobs from $\{N$-j_{q1}-

j_{q3}}, including j_{q2} and j_{q4} are assigned to (i_2, k_{i2}). This solution is perturbed by selecting a job j that is currently assigned to $i \in M_W$, and reassigning it to (i_2, k_{i2}), in place of j_{q2}. Both solutions satisfy (41) and therefore (43) exactly, as the assignment of j_{q2} is hidden. The choice of $j \in \{N\text{-}j_{q3}\text{-}j_{q4}\text{-}j_{q2}\}$ is arbitrary, as is the choice of (i, k_i) for $i \in M\text{-}i_2$ to which it is assigned to. Therefore $\pi_{i_1 jk_{i1}} = \pi_{i_2 jk_{i2}} = \pi b_j$ for each $j \in \{N\text{-}j_{q3}\text{-}j_{q4}\text{-}j_{q2}\}$ and $k_1 \le k_{i1} \le K_{i1}\text{-}1$, $1 \le k_{i2} \le k_2$. Note that this includes $\pi_{i_2 j 2} = \pi b_j$. Therefore, from the earlier result, it follows that $\pi b_j = \pi a_j$ for each $j \in \{N\text{-}j_{q3}\text{-}j_{q4}\text{-}j_{q2}\}$ and $k_1 \le k_{i1} \le K_{i1}\text{-}1$, $1 \le k_{i2} \le k_2$. Note that if (i_1, K_{i1}) is used in $(z, y)^2$, then since the assignment of j_{q4} is not hidden, it can be shown that $\pi_{i_1 jK_{i1}} = \pi_{i_2 jk_{i2}} = \pi a_j$ for each $j \in \{N\text{-}j_{q3}\text{-}j_{q2}\}$. Next, consider $(z, y)^3$: i) $y_{i_1 k_{i1}} = y_{i_2 k_{i2}} = 1$, where $k_1 \le k_{i1} \le K_{i1}\text{-}1$, $k_2\text{+}1 \le k_{i2} \le K_{i2}\text{-}1$ but $k_{i1}\text{+}k_{i2} \le n\text{-}1$, ii) $y_{ik_i} = 1$ for each $i \in M_W \subseteq M\text{-}W$ and cardinalities k_i chosen such that $\sum_{i \in M_W} k_i = n - k_{i1} - k_{i2} \ge 1$, and iii) jobs j_{q1} and j_{q3} are assigned to (i_1, k_{i1}), while j_{q2} and j_{q4} are assigned to (i_2, k_{i2}). Here, the assignment of j_{q4} is hidden, and therefore (41) and (43) are satisfied exactly. By exchanging the assignment of j_{q4} with that of a job j currently assigned to an $i \in M_W$, as well as exchanging it with j_{q3}, it is easy to show that $\pi_{i_1 jk_{i1}} = \pi_{i_2 jk_{i2}} = \pi a_j$ for each $j \in \{N\text{-}j_{q1}\text{-}j_{q2}\text{-}j_{q4}\}$ and $k_1 \le k_{i1} \le K_{i1}\text{-}1$, $k_2\text{+}1 \le k_{i2} \le K_{i2}\text{-}1$. Finally, consider $(z, y)^4$, the complement of $(z, y)^3$, where agent-cardinality combinations (i_1, k_{i1}) and (i_2, k_{i2}) are used, with $1 \le k_{i1} \le k_1\text{-}1$ and $k_2\text{+}1 \le k_{i2} \le K_{i2}\text{-}1$. In this assignment, j_{q1} is hidden. Using a perturbation similar to that in $(z, y)^3$, it can be shown that $\pi_{i_1 jk_{i1}} = \pi_{i_2 jk_{i2}} = \pi a_j$ for each $j \in \{N\text{-}j_{q1}\text{-}j_{q3}\text{-}j_{q4}\}$ and $1 \le k_{i1} \le k_1\text{-}1$, $k_2\text{+}1 \le k_{i2} \le K_{i2}\text{-}1$. Thus, from the perturbations of $(z, y)^1$, $(z, y)^2$, $(z, y)^3$ and $(z, y)^4$, we show that the coefficients of the z variables in (43) are each equal to πa_j, one for each $j \in N$.

Consider again a variant of $(z, y)^2$, wherein a set of jobs $H \subset N$, with $H_q \subset H$, are to be assigned to (i_1, k_{i1}) and (i_2, k_{i2}) alone. That is, $k_{i1}\text{+}k_{i2} = |H| = h$. Further, j_{q3} is assigned to (i_1, k_{i1}), while j_{q4} is assigned to (i_2, k_{i2}). However, jobs in $N\text{-}H$ are unassigned. For such a feasible solution, (41) reduces to

$$\sum_{j \in \{H - j_{q2} - j_{q4}\}} z_{i_1 jk_{i1}} + \sum_{j \in \{H - j_{q2} - j_{q3}\}} z_{i_2 jk_{i2}} \le h - 3 \qquad (44)$$

The corresponding constraint (43) becomes

$$\sum_{j\in\{H-j_{q2}-j_{q4}\}} \pi a_j z_{i_1 j k_{i1}} + \sum_{j\in\{H-j_{q2}-j_{q3}\}} \pi a_j z_{i_2 j k_{i2}} \leq C \tag{45}$$

where C denotes the composite value obtained after fixing the values of y and z as described above. Specifically,

$$C = \pi_{i_1 k_{i1}} + \pi_{i_2 k_{i2}} - \pi a_{j_{q3}} - \pi a_{j_{q4}} + \pi_0 \tag{46}$$

By definition, every feasible solution that satisfies (44) exactly, satisfies (45) exactly as well. There are h-3 linearly independent selections of k_{i1} jobs from the set $\{H\text{-}j_{q2}\text{-}j_{q4}\}$. Consequently, there are h-3 linearly independent solutions that satisfy both (44) and (45) exactly, implying that (45) is a scalar multiple of (44). Thus, $\pi a_j = \pi a$ for each $j\in\{H\text{-}j_{q2}\text{-}j_{q3}\text{-}j_{q4}\}$. Since the selection of H-H_q from N-H_q is arbitrary, it follows that $\pi a_j = \pi a$ for each $j\in\{N\text{-}H_q\}$. In addition, $\pi a_{jq1} = \pi a$.

As was done with $(z, y)^2$ above, starting with $(z, y)^1$, $(z, y)^3$ and $(z, y)^4$, where $H\subset N$ jobs are assigned to (i_1, k_{i1}) and (i_2, k_{i2}), including jobs in H_q, it can be shown that $\pi a_{jq3} = \pi a$, $\pi a_{jq2} = \pi a$ and $\pi a_{jq4} = \pi a$, respectively. Further, $C = (h-3)\pi a$.

Consider another variant of $(z, y)^2$, where in addition to a designated set of jobs in H assigned to (i_1, k_{i1}) and (i_2, k_{i2}), k_i>0 jobs in N-H are assigned to agent-cardinality (i, k_i) for some $i\in M$-W. As a result, the left-hand-side of (45) increases by $k_i \pi a$. This is so for any $i\in M$-W and $1\leq k_i \leq K_i$. Therefore, $\pi_{ik_i} = k_i \pi a$ for any $i\in M$-W and $1\leq k_i\leq K_i$. Now consider a solution $(z, y)^5$, where agent-cardinality (i_1, k_{i1}) alone is used, where $1\leq k_{i1}\leq K_{i1}$. In addition, all k_{i1} jobs that are assigned to i_1 come from the set N-H_q. Such an assignment is certainly possible as $K_{i1} \leq n$-4. This solution satisfies (41) exactly as there are no hidden assignments. Therefore, $\pi_{i_1} k_{i1} + \pi_0 = k_{i1}\pi a$ for all $1\leq k_{i1}\leq K_{i1}$. In a similar fashion, it can be shown that $\pi_{i_2} k_{i2} + \pi_0 = k_{i2}\pi a$ for all $1\leq k_{i2}\leq K_{i2}$. We now refer back to $(z, y)^2$, where the set of jobs H, with $H_q\subset H$, are assigned to (i_1, k_{i1}) and (i_2, k_{i2}) alone, with $k_{i1}+k_{i2} = h$, j_{q3} assigned to (i_1, k_{i1}) and j_{q4} is assigned to (i_2, k_{i2}). Here, $\pi_{i_1} k_{i1} + \pi_{i_2} k_{i2} + \pi_0 = (k_{i1} + k_{i2} - 1)\pi a$. From the three equations, we get $\pi_0 = \pi a$, $\pi_{i_1 k_{i1}} = (k_{i1} - 1)\pi a$ and

$\pi_{i_2 k_{i2}} = (k_{i2} - 1)\pi a$. This concludes the argument that (43) is a scalar multiple of (41) and therefore (41) non-trivial facet of $H_4(d)^{\leq}$. □

Theorem 13 The 2-Agent Cardinality Matching Inequality (41) is a non-trivial facet of $HU(d)^{\leq}$.

Proof: Starting with (41), a sequential lifting procedure to increase the coefficient values of variables $z_{ijk} \in \{\boldsymbol{z^c}\text{-}z^r\}$. The resulting inequality obtained will be a facet of $HU(d)^{\leq}$. It therefore suffices to show that after lifting, the coefficients of variables in $\{\boldsymbol{z^c}\text{-}\boldsymbol{z^r}\}$ all remain zero.

Let U_z denote the index set of all z variables in $\{\boldsymbol{z^c}\text{-}z^r\}$, while $V_z \subseteq U_z$ represents those whose coefficients have already been lifted. At stage l of the sequential lifting procedure, the coefficient of $z_{ijk}(l)$ for $l \in \{U_z\text{-}V_z\}$ is lifted by solving the optimization problem:

$$\pi_{ijk}(l) = Min\{1+\pi_{yc}y^c\text{-}z^r\text{-}\pi_r(V_z)z_r(V_z)|\ (5^{\leq}), (6), (7), (8), (9), z_{ijk}(l) = 1, z_{ijk}(s) = 0, \forall s \in \{U_z\text{-}V_z\}\}, \quad (47)$$

where $(1+\pi_{yc}y^c)$ represents the right-hand-side of (41), z^r the left-hand-side of (41) and $\pi_r(V_z)$ the lifted coefficients of variables in V_z till iteration l-1. Observe that for the lifted inequality to remain valid, $\pi_{ijk}(l) \geq 0$. Therefore, in each iteration l, it suffices to identify a feasible solution to (47) whose objective function value is zero.

Consider first the sequence of variables $z_{i_1 j_{q1} k_{i1}}(l)$, $l \in \{U_z\text{-}V_z\}$, for some $1 \leq k_{i1} \leq k_1$-1. Starting with l=1, with say $k_{i1}=k_1$-1, the optimal solution to (47) is, $y_{i_1 k_{i1}} = y_{i_2 k_2 +1} = 1$, and the rest of the y variables set to zero. First, (k_{i1}-1) jobs from N-$\{j_{q2}, j_{q3}\}$, including j_{q4}, are assigned to agent-cardinality (i_1, k_{i1}). Note that j_{q1} is assigned to (i_1, k_{i1}). Next, from the remaining unassigned jobs, (k_2+1) jobs, which includes j_{q2} and j_{q3} are assigned to (i_2, k_2+1). This results in $\pi_{i_1 j k_1}(1) = 0$. In subsequent iterations, for other values of $k_{i1} \geq 2$, the same lifting solution gives $\pi_{i_1 j_{q1} k_{i1}}(l) = 0$. For k_{i1}=1, agent-cardinality (i_2, K_{i2}) is used in place of (i_2, k_2+1). Here, j_{q4} as well is assigned to (i_2, K_{i2}), resulting in $\pi_{i_1 j_{q1},1}(l) = 0$. Consider next the lifting of coefficients of $z_{i_2 j_{q1} k_{i2}}(l)$, for $k_{i2} \geq k_2$+1. Here, the optimal solution is $y_{i_1 1} = y_{i_2 k_{i2}} = 1$, with the rest of the y variables set to zero. Job j_{q4} is assigned to (i_1, 1), while (k_{i2}-1) jobs from N-j_{q4}, including $\{j_{q1}, j_{q2}, j_{q3}\}$ are assigned to (i_2, j_{i2}), resulting in $\pi_{i_2 j_{q1} k_{i2}}(l) = 0$, for all $k_2+1 \le k_{i2} \le K_{i2}$. Finally, associated with j_{q1}, consider lifting of coefficients of $z_{ij_{q1}k_i}(l)$, for some $i \in M\text{-}W$. Here, the optimal solution is $y_{i_1 1} = y_{i_2 k_2+1} = y_{ik_i} = 1$. While j_{q1} is assigned to (i, k_i), for $i \in M\text{-}W$, j_{q4} is assigned to $(i_1, 1)$ and $\{j_{q2}, j_{q3}\}$ assigned to (i_2, j_{k2+1}). The rest of the jobs are selected from $N\text{-}H_q$ to match the cardinality requirement. This ensures that $\pi_{ij_{q1}k_i}(l)$=0 for all $i \in M\text{-}W$ and $1 \le k_i \le K_i$.

For the sequence of variables, $z_{i_1 j_{q2} k_{i1}}(l)$ for $k_1 \le k_{i1} \le K_{i1}$, $z_{i_2 j_{q2} k_{i2}}(l)$ for $1 \le k_{i2} \le k_2$ and $z_{ij_{q2}k_1}(l)$ for $i \in M\text{-}W$ and $1 \le k_i \le K_i$, it is easy to show that the corresponding coefficients are all zero. The lifting of the coefficients involving assignment of j_{q2} is a complement of the assignment of j_{q1} described earlier. Now consider the sequence of variables, $z_{i_1 j_{q3} k_{i1}}(l)$, $l \in \{U_z\text{-}V_z\}$, for some $1 \le k_{i1} \le k_1$-1. Here, the optimal solution is, $y_{i_1 k_{i1}} = y_{i_2 2} = 1$, and the rest the y variables set to zero. Jobs j_{q3} is of course is assigned to (i_1, k_{i1}). The rest of the $(k_{i1}$-1) jobs are drawn from $N\text{-}H_q$. Similarly, two jobs from $N\text{-}H_q$ are assigned to $(i_2, 2)$. Clearly, this solution results in $\pi_{i_1 j_{q3} k_{i1}}(l)$=0 for $1 \le k_{i1} \le k_1$-1. For the sequence of variables, $z_{i_2 j_{q3} k_{i2}}(l)$, $l \in \{U_z\text{-}V_z\}$, $1 \le k_{i2} \le k_2$, the optimal solution is, $y_{i_1 2} = y_{i_2 k_{i2}} = 1$, and the rest the y variables set to zero. Here, j_{q3} is assigned to (i_2, k_{i2}), while the remaining available slots for (i_2, k_{i2}) and $(i_1, 2)$ are filled up by jobs from $N\text{-}H_q$. This results in $\pi_{i_2 j_{q3} k_{i2}}(l)$ =0 for $1 \le k_{i2} \le k_2$. For lifting of coefficients of variables, $z_{ij_{q3}k_i}(l)$, the optimal solution used is $y_{i_1 1} = y_{i_2 2} = y_{ik_i} = 1$, and the rest of the y variables set to zero. Here, j_{q1} is assigned to $(i_1, 1)$, j_{q2} and j_{q4} are assigned to $(i_2, 2)$, while j_{q3} and $(k_i$-1) jobs from $N\text{-}H_q$ are assigned to (i, k_i). This results in $\pi_{ij_{q3}k_i}(l)$ =0 for all $i \in M\text{-}W$ and $1 \le k_i \le K_i$.

Consider next obtaining $\pi_{i_1 j_{q4} k_{i1}}(l)$ for $k_1 \le k_{i1} \le K_{i1}$-1. The optimal solution to (47) is, $y_{i_1 k_{i1}} = y_{i_2 k_2+1} = 1$ with j_{q2} and j_{q4} assigned to (i_1, k_{i1}), j_{q1} and j_{q3} assigned to (i_2, k_2+1), while the rest of the $(k_{i1}+k_2$-3) jobs are drawn from $N\text{-}H_q$ and assigned to (i_1, k_{i1}) and (i_2, k_2+1), consistent with their cardinalities. This results in $\pi_{i_1 j_{q4} k_{i1}}(l)$=0 for $k_1 \le k_{i1} \le K_{i1}$-1. Similarly, for obtaining $\pi_{i_2 j_{q4} k_{i2}}(l)$ for $k_2+1 \le k_{i2} \le K_{i2}$-1, the optimal solution used is, $y_{i_1 k_1} = y_{i_2 k_{i2}} = 1$, with j_{q2} and j_{q3} assigned to (i_1, k_1),

j_{q1} and j_{q4} assigned to (i_2, k_{i2}), and (k_1+k_{i2}-4) jobs from N-H_q assigned to (i_1, k_1) and (i_2, k_{i2}), consistent with their cardinalities. Thus, $\pi_{i_2 j_{q4} k_{i2}}(l)$=0 for $k_2+1 \le k_{i2} \le K_{i2}-1$. Finally, it can be shown that $\pi_{ij_{q4}k_i}(l)$=0, for all $i \in M$-W and $1 \le k_i \le K_i$, by using the optimal solution, $y_{i_1k_1} = y_{i_2k_2+1} = y_{ik_i} = 1$, in which j_{q4} is assigned to (i,k_i), j_{q2} is assigned to (i_1, k_1), j_{q1} and j_{q3} assigned to (i_2, k_2+1), and (k_1+k_2+k_i-3) jobs from N-H_q assigned to (i_1, k_1), (i_2, k_2+1) and (i,k_i), respectively. Thus, the coefficients of all variables in {$\boldsymbol{z^c}$-$z^{\boldsymbol{r}}$} remain zero after lifting.

□

5.2 Lifting of 2-Agent Cardinality Matching Inequality for $H(d)^{\le}$

While the 2-Agent Cardinality Matching inequality is a non-trivial facet of $HU(d)^{\le}$, it need not be for $H(d)^{\le}$ (defined in (17)), due to the presence of knapsack constraints (4). We now show how (41) can be lifted to become a non-trivial facet of $H(d)^{\le}$. A sequential lifting procedure is employed for each of the coefficients of the missing z variables in (41). The optimization problem that needs to be solved is essentially the same as that in (47), except that the knapsack constraints (4) have to be satisfied as well. With the inclusion of (4), the sequential lifting procedure boils down to solving a bin packing problem for each coefficient.

To begin with, consider the missing z variables associated with j_{q1}. To determine $\pi_{i_1 j_{q1} k_{i1}}(l)$, for $1 \le k_{i1} \le k_1-1$, we determine a set $N(i_1,k_{i1}-1) \subset \{N-j_{q1}-j_{q3}\}$, which consists of the first k_{i1}-1 jobs in {N-j_{q1}-j_{q3}} after sorting them in non-decreasing order in terms of $a_{i1,j}$. If the assignment of jobs in $N(i_1,k_{i1}-1)$ and j_{q1} to (i_1, k_{i1}) satisfies (4), then the optimal solution is: $y_{i_1k_{i1}} = 1$ and $z_{i_1jk_{i1}} = 1$ for all $j \in \{N(i_1,k_{i1}-1), j_{q1}\}$, and $y_{i_21} = 1$, $z_{i_1j_{q4}1} = 1$ since $a_{ij} \le b_i$ for all $i \in M$, $j \in N$. The rest of the y and z variables are set to zero. This results in, $\pi_{i_1 j_{q1} k_{i1}}(l)$ = (k_{i1}-1)+0+1-(k_{i1}-1)+1 = 0. If the assignment of $N(i_1,k_{i1}-1)$ and j_{q1} to (i_1, k_{i1}) violates (4), then $N(i_1,k_{i1}-1)$ is modified by replacing the largest job in it by j_{q3}. If the resulting solution is feasible, then since j_{q3} is hidden from (i_1, k_{i1}) as well, $\pi_{i_1 j_{q1} k_{i1}}(l)$=1. However, if even with j_{q3}, (4) is violated, then the formulation itself can be strengthened by setting $z_{i_1 j_{q1} k_{i1}} = 0$. Essentially, the same steps are used to determine $\pi_{i_2 j_{q1} k_{i2}}(l)$, where k_2+1≤k_{i2}≤K_{i2}, except that $N(i_2,k_{i2}$-1)⊂{N-j_{q1}-j_{q4}} is assigned initially. If infeasible, then $N(i_2,k_{i2}$-1) is constructed by including j_{q4}. Thus, here as well, $\pi_{i_2 j_{q1} k_{i2}}(l)$ is either 0 or 1, or that $z_{i_2 j_{q1} k_{i2}} = 0$. To determine $\pi_{ij_{q1}k_i}(l)$ where i∈M-W and 1≤k_i≤K_i, the set $N(i,k_i$-1)⊂{N-H_q}, consisting of the first k_i-1 jobs in N-H_q sorted in non-decreasing order in terms of a_{ij} is obtained. If $N(i,k_i$-1) and j_{q1}, upon being assigned to (i, k_i) satisfy (4), then the solution $y_{i_1 k_i} = 1$ and $z_{i_1 j k_i} = 1$ for all j∈{ $N(i,k_i$-1), j_{q1}}, $y_{i_1 1} = 1$, $z_{i_1 j_{q4} 1} = 1$, $y_{i_2 1} = 1$, $z_{i_2 j_{q2} 1} = 1$, and the rest of the y and z variables set to zero, results in $\pi_{ij_{q1}k_i}(l)$=0. If the above assignment is infeasible, then we consider assignments where one job from the set {j_{q2}, j_{q3}, j_{q4}} is included in $N(i,k_i$-1). If a feasible assignment exists then $\pi_{ij_{q1}k_i}(l)$=1. If no feasible assignment exists above, then solutions in which two jobs from the set {j_{q2}, j_{q3}, j_{q4}} are included in $N(i,k_i$-1). If a feasible assignment is found, then this results in $\pi_{ij_{q1}k_i}(l)$ =2. If not, an assignment in which all three jobs in {j_{q2}, j_{q3}, j_{q4}} being included in $N(i,k_i$-1) is explored. If feasible, then $\pi_{ij_{q1}k_i}(l)$=3, else the formulation is strengthened by setting $z_{ij_{q1}k_i}$=0.

The process of lifting of coefficients that correspond to missing variables associated with j_{q2} mirror in a complementary way those associated with j_{q1} described above and therefore need no elaboration. In the case of j_{q3}, we begin with the determination of $\pi_{i_1 j_{q3} k_{i1}}(l)$, for 1≤$k_{i1}$≤$k_1$-1. Here, we first determine the set $N(i_1,k_{i1}$-1)⊂{N-j_{q1}-j_{q3}}, as was done to determine $\pi_{i_1 j_{q1} k_{i1}}(l)$. If the assignment of $N(i_1,k_{i1}$-1) and j_{q3} to (i_1, k_{i1}) satisfies (4), then $\pi_{i_1 j_{q3} k_{i1}}(l)$=0, else we consider modifying the set $N(i_1,k_{i1}$-1) to include j_{q1}. If such an assignment is found to be feasible, then $\pi_{i_1 j_{q3} k_{i1}}(l)$=1, else the formulation is strengthened by setting $z_{i_1 j_{q3} k_{i1}}$=0. The process of determining $\pi_{i_1 j_{q3} k_{i2}}(l)$, for 1≤$k_{i2}$≤$k_2$ is identical to that for $\pi_{i_1 j_{q3} k_{i1}}(l)$. The process of determining $\pi_{ij_{q3}k_i}(l)$ closely mirrors $\pi_{ij_{q1}k_i}(l)$ described earlier. The difference is that initially, j_{q3} along with $N(i,k_i$-1) is assigned to (i, k_i) instead of j_{q1}. If found feasible, $\pi_{ij_{q3}k_i}(l)$=0. If infeasible, then in subsequent steps, the inclusion of additional jobs into $N(i,k_i$-1) is done from the set {j_{q1}, j_{q2}, j_{q4}}. Thus, here as well, in subsequent steps, $\pi_{ij_{q3}k_i}(l)$ can take one of the three values: 1, 2 or 3, or that the constraint $z_{ij_{q3}k_i}$ = 0 is added to the formulation.

Finally, consider lifting of coefficients of missing variables associated with j_{q4}. To determine $\pi_{i_1 j_{q4} k_{i1}}(l)$ for $k_1 \le k_{i1} \le K_{i1}$, initially the set $N(i_1,k_{i1}\text{-}1) \subset \{N\text{-}j_{q2}\text{-}j_{q4}\}$ is determined to check if it along with j_{q4} can satisfy (4), if assigned to (i_1, k_{i1}). If so, then by assigning j_{q2} to (i_2, 1) as well, an optimal solution is obtained with $\pi_{i_1 j_{q4} k_{i1}}(l) = 0$. If such an assignment is not possible, then the set $N(i_1,k_{i1}\text{-}1)$ is modified by including j_{q2} in it. If, as a result, a feasible solution is found, then $\pi_{i_1 j_{q4} k_{i1}}(l)$ =1, otherwise the constraint $z_{i_1 j_{q4} k_{i1}}$ =0 is added to the formulation. The process of determining $\pi_{i_2 j_{q4} k_{i2}}(l)$ for $k_2+1 \le k_{i2} \le K_{i2}$, exactly mirrors that used for $\pi_{i_1 j_{q4} k_{i1}}(l)$ and therefore needs no elaboration. Similarly, the process of determining $\pi_{ij_{q4}k_i}(l)$ where $i \in M\text{-}W$ and $1 \le k_i \le K_i$, closely mirrors the process used for determining $\pi_{ij_{q1}k_i}(l)$, $\pi_{ij_{q2}k_i}(l)$ or $\pi_{ij_{q3}k_i}(l)$. Thus, $\pi_{ij_{q4}k_i}(l)$ can take one of four values: 0, 1, 2, 3, or that $z_{ij_{q4}k_i}$ =0.

It needs to be emphasized that the optimization used in the sequential lifting procedure are all easy, since each one of them involve solving a bin-packing problem with one bin. Thus, without much computational effort, the 2-Agent Cardinality matching inequality (41) can lifted to obtain an inequality that is a facet of $H(d)^{\le}$.

5.3 The 2-Agent Cardinality Matching Inequality when *m* = 2

We now consider the special case of (GAP) consisting of just two agents. If any one agent can accommodate all the jobs, then (4) becomes redundant and the problem becomes trivial. Hence, we assume that both agents are needed to accommodate all the jobs. Thus for instances where $M = \{i_1, i_2\}$, the 2-Agent Cardinality Matching Inequality (41) can be further strengthened to become

$$\sum_{j\in\{N-j_{q1}\}}\sum_{k_{i1}=n-K_{i2}}^{k_1-1} z_{i_1 jk_{i1}} + \sum_{j\in\{N-j_{q2}\}}\sum_{k_{i1}=k_1}^{K_{i1}} z_{i_1 jk_{i1}} + \sum_{j\in\{N-j_{q2}\}}\sum_{k_{i2}=n-K_{i1}}^{k_2} z_{i_2 jk_{i2}} + \sum_{j\in\{N-j_{q1}\}}\sum_{k_{i2}=k_2+1}^{K_{i2}} z_{i_2 jk_{i2}} \le \sum_{k_{i1}=n-K_{i2}}^{K_{i1}} (k_{i1}-1) y_{i_1 k_{i1}} + \sum_{k_{i2}=n-K_{i1}}^{K_{i2}} (k_{i2}-1) y_{i_2 k_{i2}} + 1. \quad (48)$$

Observe that when $M = \{i_1, i_2\}$, the use of agent-cardinalities (i_1, k_{i1}) and (i_2, k_{i2}) should be such that $k_{i1}+k_{i2} = n$. Consequently, the smallest feasible cardinality associated with i_1 is n-K_{i2}, and with i_2, n-K_{i1}. This also rules out the possibilities, i) $k_{i1} \leq k_1$-1 and $k_{i2} \leq k_2$, and ii) $k_{i1} \geq k_1$ and $k_{i2} \leq k_2$+1. In the former case, the need to define j_{q3}, as a job whose assignment is hidden from (i_1, k_{i1}) and (i_2, k_{i2}) is no longer required. Similarly, in the latter case, the need to define j_{q4} is unnecessary.

Since (48) is a special case of (41), it is a non-trivial facet of $HU(d)^{\leq}$ when m=2. Note that starting from Proposition 11, where $m = 2$, the dimension of $HU(d)^{\leq}$ becomes $2n(K_{i1}+K_{i2}-n)$, and therefore the dimension of the hyperplane (48) is $2n(K_{i1}+K_{i2}-n)$-1. Since every *SOS* constraint $(5)^{\leq}$ is a trivial facet of $HU(d)^{\leq}$, the dimension of $HU(d)$ is $2n(K_{i1}+K_{i2}-n)-n$.

Theorem 14: Consider the polytope $HU(d)$ defined by agents, $M = \{i_1, i_2\}$ and a set N consisting of a finite number of jobs. The 2-Agent Cardinality Matching Inequality (48), along with trivial facets, a) $x_{ijk} \geq 0$, b) $z_{ijk} \leq y_{ik}$, and c) constraints (7), as described in Proposition 12, completely describe the polytope $HU(d)$.

Proof: Let $LPU(d) = \{(z, y) \in R^p | (z, y)$ satisfies (5)-(8), $z \geq 0$, $y \geq 0\}$, $p = 2(n+1)(K_{i1}+K_{i2}-n)\}$, the LP relaxation of (**Pd**) without the knapsack constraints (4). Clearly, every extreme point of $HU(d)$ is also an extreme point of $LPU(d)$. However, in addition, $LPU(d)$ consists of extreme points which are fractional in nature. Every extreme point of $HU(d)$ is characterized by the following: $y_{i_1 k_{i1}} = y_{i_2 k_{i2}} = 1$, $k_{i1}+k_{i2} = n$, $z_{i_1 j k_{i1}} = 1$ for all $j \in N_{k_{i1}}$, $z_{i_2 j k_{i2}} = 1$ for all $j \in N_{k_{i2}}$, where $N_{k_{i1}} \cup N_{k_{i2}} = N$, $N_{k_{i1}} \cap N_{k_{i2}} = \phi$, $|N_{k_{i1}}| = k_{i1}$ and $|N_{k_{i2}}| = k_{i2}$. Given this structure, there are two possible sets of solutions. In one set, $k_{i1} \leq k_1$-1 and $k_{i2} \geq k_2$+1, while in the other, $k_{i1} \geq k_1$ and $k_{i2} \leq k_2$.

Each inequality in (48) is uniquely defined by k_1, k_2 and a selection of j_{q1} and j_{q2} from N. It is indeed easy to verify that every inequality in (48) passes through all the extreme points of $HU(d)$, i.e., every extreme point of $HU(d)$ satisfies (48) as an inequality. This is indeed possible since as

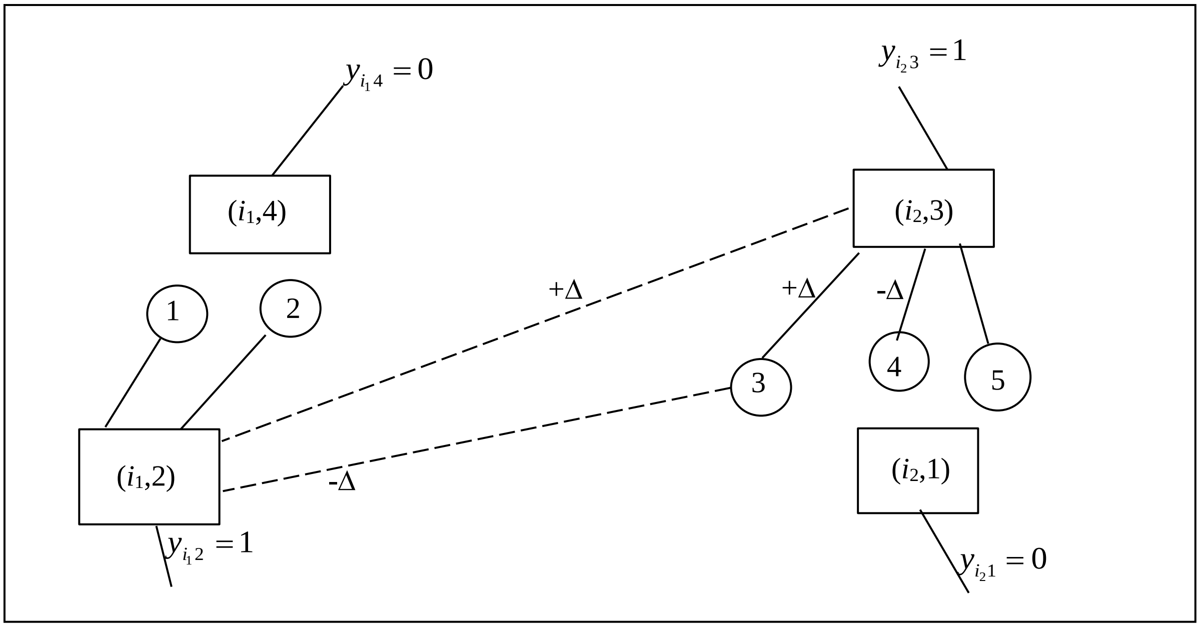

Figure 5: Illustration of movement from one integer solution to an adjacent integer solution.

observed earlier, the dimension of $HU(d)$ is $2n(K_{i1}+K_{i2}-n)-n$, while that of (48) is $2n(K_{i1}+K_{i2}-n)-1$.

Given the above, it suffices to show that (48) violates every fractional extreme point of $LPU(d)$. In turn, since (48) passes through every extreme point of $HU(d)$, it is sufficient to identify fractional extreme points of $LPU(d)$ that are adjacent to integer extreme points, and show that they violate (48).

Note that, including slack variables associated with (7) and (8), $LPU(d)$ is defined by a total of $2(2n+1)(K_{i1}+K_{i2}-n)+2$ variables and $2(n+1)(K_{i1}+K_{i2}-n)+n+2$ constraints. However, since any integer extreme point of $LPU(d)$ consists of exactly n non-zero variables, it is highly degenerate. To explore adjacent extreme points, one perturbation of an integer extreme point of $LPU(d)$ is to increase the value of the non-basic variable $z_{i_2 jk_{i2}}$ by Δ>0, where $j \in N_{k_{i1}}$ and therefore $z_{i_1 jk_{i1}} = 1$. To satisfy (5), $z_{i_1 jk_{i1}}$ is decreased by Δ. To satisfy the cardinality constraints at (i_1, k_{i1}) and (i_2, k_{i2}), a $j' \in N_{k_{i2}}$ is chosen wherein, $z_{i_2 j' k_{i2}} = 1 - \Delta$, while $z_{i_1 j' k_{i1}} = \Delta$. This is illustrated in Figure 5, where *j*=2 and *j'*=3. In the limiting case, this perturbation ends with Δ=1, at which point $z_{i_2 j' k_{i2}}$ becomes non-basic. The resulting extreme point solution is also integer wherein there is an exchange of assignments of *j* and *j'*. Thus, in this perturbation, $y_{i_1 k_{i1}} = y_{i_2 k_{i2}} = 1$ remains.

Another perturbation to explore adjacent extreme points is to increase the value of $y_{i_1 k'_{i1}}$ where $k'_{i1} \neq k_{i1}$ given that $y_{i_1 k_{i1}} = 1$. Alternately, $y_{i_2 k'_{i2}}$ can be chosen for perturbation visa-vis $y_{i_2 k_{i2}}$ in the same way. Here, the value of $y_{i_1 k'_{i1}}$, which is currently zero is increased it by Δ. Simultaneously, the value of $y_{i_2 k'_{i2}}$ where $k'_{i2} = n - k'_{i1}$ is also increased by Δ. Naturally, to satisfy (7), the values of $y_{i_1 k_{i1}}$ and $y_{i_2 k_{i2}}$ are decreased by Δ. Let $N_{k'_{i1}}$ and $N_{k'_{i2}}$ be defined such that, i) $|N_{k'_{i1}}| = k'_{i1}$ and $|N_{k'_{i2}}| = k'_{i2}$, ii) $N_{k'_{i1}} \cup N_{k'_{i2}} = N$ and iii) $N_{k'_{i1}} \cap N_{k'_{i2}} = \phi$. We now let i) $z_{i_1 j k'_{i1}} = \Delta$ for each $j \in N_{k'_{i1}}$, ii) $z_{i_2 j k'_{i2}} = \Delta$ for each $j \in N_{k'_{i2}}$, iii) $z_{i_1 j k_{i1}} = 1 - \Delta$ for each $j \in N_{k_{i1}}$ and iv) $z_{i_2 j k_{i2}} = 1 - \Delta$ for each $j \in N_{k_{i2}}$. Such a perturbation is certainly possible as constraints (5), (6) and (7) are satisfied exactly. Here as well, an adjacent extreme point is found when Δ = 1, which is also integer, with $y_{i_1 k'_{i1}} = y_{i_2 k'_{i2}} = 1$ with jobs in $N_{k'_{i1}}$ and $N_{k'_{i2}}$ assigned to (i_1, k'_{i1}) and (i_2, k'_{i2}) respectively.

The only other perturbation that is possible is identical to the previous one, except that $N_{k'_{i1}} \cap N_{k'_{i2}} \neq \phi$. Therefore, $|N_{k'_{i1}} \cup N_{k'_{i2}}| < n$. Let $N_{k'_{i1} k'_{i2}} = N_{k'_{i1}} \cap N_{k'_{i2}}$, while $N_{k'_{i1} k'_{i2} k_{i1}} = N_{k'_{i1} k'_{i2}} \cap N_{k_{i1}}$ and $N_{k'_{i1} k'_{i2} k_{i2}} = N_{k'_{i1} k'_{i2}} \cap N_{k_{i2}}$. Similarly, let $\overline{N}_{k'_{i1} k'_{i2}} = N - \{N_{k'_{i1}} \cup N_{k'_{i2}}\}$, while $\overline{N}_{k'_{i1} k'_{i2} k_{i1}} = \overline{N}_{k'_{i1} k'_{i2}} \cap N_{k_{i1}}$ and $\overline{N}_{k'_{i1} k'_{i2} k_{i2}} = \overline{N}_{k'_{i1} k'_{i2}} \cap N_{k_{i2}}$. Clearly, $N_{k'_{i1} k'_{i2}} = N_{k'_{i1} k'_{i2} k_{i1}} \cup N_{k'_{i1} k'_{i2} k_{i2}}$, $\overline{N}_{k'_{i1} k'_{i2}} = \overline{N}_{k'_{i1} k'_{i2} k_{i1}} \cup \overline{N}_{k'_{i1} k'_{i2} k_{i2}}$ and $|N_{k'_{i1} k'_{i2}}| = |\overline{N}_{k'_{i1} k'_{i2}}|$. Here, the integer solution $y_{i_1 k_{i1}} = y_{i_2 k_{i2}} = 1$, $z_{i_1 j k_{i1}} = 1$ for all $j \in N_{k_{i1}}$, $z_{i_2 j k_{i2}} = 1$ for all $j \in N_{k_{i2}}$ is perturbed in part as follows: i) $y_{i_1 k'_{i1}} = \Delta$, $z_{i_1 j k'_{i1}} = \Delta$ for each $j \in N_{k'_{i1}}$, ii) $y_{i_2 k'_{i2}} = \Delta$, $z_{i_2 j k'_{i2}} = \Delta$ for each $j \in N_{k'_{i2}}$ and iii) $y_{i_1 k_{i1}} = y_{i_2 k_{i2}} = 1 - \Delta$. Note that the perturbation described in i) and ii) satisfy (6) and (8) exactly. What remains is to specify the values of $z_{i_1 j k_{i1}}$ for each $j \in N_{k_{i1}}$ and $z_{i_2 j k_{i2}}$ for each $j \in N_{k_{i2}}$ such that (5), (6) and (8) are satisfied.

Since $z_{i_1jk'_{i1}} = \Delta$ and $z_{i_2jk'_{i2}} = \Delta$ for each $j \in N_{k'_{i1}k'_{i2}}$, it follows that to satisfy (5), $z_{i_1jk_{i1}} = 1-2\Delta$ and $z_{i_2jk_{i2}} = 1-2\Delta$, for each $j \in N_{k'_{i1}k'_{i2}k_{i1}}$ and $j \in N_{k'_{i1}k'_{i2}k_{i2}}$, respectively. Further, to satisfy (5) and (8), $z_{i_1jk_{i1}} = 1-\Delta$, $z_{i_2jk_{i2}} = \Delta$ for every $j \in \overline{N}_{k'_{i1}k'_{i2}k_{i1}}$, and $z_{i_1jk_{i1}} = \Delta$, $z_{i_2jk_{i2}} = 1-\Delta$ for every $j \in \overline{N}_{k'_{i1}k'_{i2}k_{i2}}$. By definition, $|N_{k'_{i1}k'_{i2}k_{i1}}| - |\overline{N}_{k'_{i1}k'_{i2}k_{i2}}| = |\overline{N}_{k'_{i1}k'_{i2}k_{i1}}| - |N_{k'_{i1}k'_{i2}k_{i2}}| = DIFF$. Thus at ($i_1$, k_{i1}), the cumulative value of z variables accounted for from $N_{k'_{i1}k'_{i2}}$ and $\overline{N}_{k'_{i1}k'_{i2}}$ is $|N_{k'_{i1}k'_{i2}k_{i1}}|(1-2\Delta) + |\overline{N}_{k'_{i1}k'_{i2}k_{i1}}|(1-\Delta) + |\overline{N}_{k'_{i1}k'_{i2}k_{i2}}|\Delta = (|N_{k'_{i1}k'_{i2}k_{i1}}| + |\overline{N}_{k'_{i1}k'_{i2}k_{i1}}|)(1-\Delta) - DIFF*\Delta$, while at ($i_2$, k_{i2}) it is $DIFF*\Delta + (|N_{k'_{i1}k'_{i2}k_{i2}}| + |\overline{N}_{k'_{i1}k'_{i2}k_{i2}}|)(1-\Delta)$. If $DIFF > 0$, then an arbitrary set $N_{Diff} \subseteq \{N_{k_{i2}} - N_{k'_{i1}k'_{i2}k_{i2}} - \overline{N}_{k'_{i1}k'_{i2}k_{i2}}\}$ is selected with $|N_{Diff}| = DIFF$. We now set $z_{i_2jk_{i2}} = 1-2\Delta$ and $z_{i_1jk_{i1}} = \Delta$ for each $j \in N_{Diff}$, $z_{i_2jk_{i2}} = 1-\Delta$ for each $j \in \{N_{k_{i2}} - N_{k'_{i1}k'_{i2}k_{i2}} - \overline{N}_{k'_{i1}k'_{i2}k_{i2}} - N_{Diff}\}$, and $z_{i_1jk_{i1}} = 1-\Delta$ for each $j \in \{N_{k_{i1}} - N_{k'_{i1}k'_{i2}k_{i1}} - \overline{N}_{k'_{i1}k'_{i2}k_{i1}}\}$. Conversely if $DIFF < 0$, then $N_{Diff} \subseteq \{N_{k_{i1}} - N_{k'_{i1}k'_{i2}k_{i1}} - \overline{N}_{k'_{i1}k'_{i2}k_{i1}}\}$ is selected such that $|N_{Diff}| = -DIFF$, and we set $z_{i_1jk_{i1}} = 1-2\Delta$ and $z_{i_2jk_{i2}} = \Delta$ for each $j \in N_{Diff}$, $z_{i_1jk_{i1}} = 1-\Delta$ for each $j \in \{N_{k_{i1}} - N_{k'_{i1}k'_{i2}k_{i1}} - \overline{N}_{k'_{i1}k'_{i2}k_{i1}} - N_{Diff}\}$, and $z_{i_2jk_{i2}} = 1-\Delta$ for each $j \in \{N_{k_{i2}} - N_{k'_{i1}k'_{i2}k_{i2}} - \overline{N}_{k'_{i1}k'_{i2}k_{i2}}\}$. Such a perturbation satisfies (5) at each $j \in N_{Diff}$, and (6) exactly, both at (i_1, k_{i1}) and (i_2, k_{i2}). This perturbation is suitably illustrated in Figure 6 below.

In the illustration above, $k_{i1} = 3$, $k'_{i1} = 4$, $k_{i2} = 3$, $k'_{i2} = 2$, $N_{k'_{i1}k'_{i2}k_{i1}} = \{3\}$, $N_{k'_{i1}k'_{i2}k_{i2}} = \{4\}$, $\overline{N}_{k'_{i1}k'_{i2}k_{i1}} = \phi$, $\overline{N}_{k'_{i1}k'_{i2}k_{i2}} = \{5, 6\}$. Given the above, $DIFF = -1$ and $N_{Diff} = \{2\}$. It is evident from this illustration that when $\Delta = 0.5$, an adjacent extreme point is found that is fractional. It is worth noting at when $\Delta = 0.5$, then any node in $N_{k'_{i1}k'_{i2}}$ (nodes 3 and 4 in Figure 6) can be chosen as j_{q1}, while any node in $\overline{N}_{k'_{i1}k'_{i2}}$ (nodes 5 and 6 in Figure 6) can be chosen j_{q2}. Clearly, the fractional solution is such that j_{q1} is hidden from (i_1, k'_{i1}) and (i_2, k'_{i2}), while j_{q2} is hidden from (i_1, k_{i1}) and (i_2, k_{i2}). This ensures that it violates (48). □

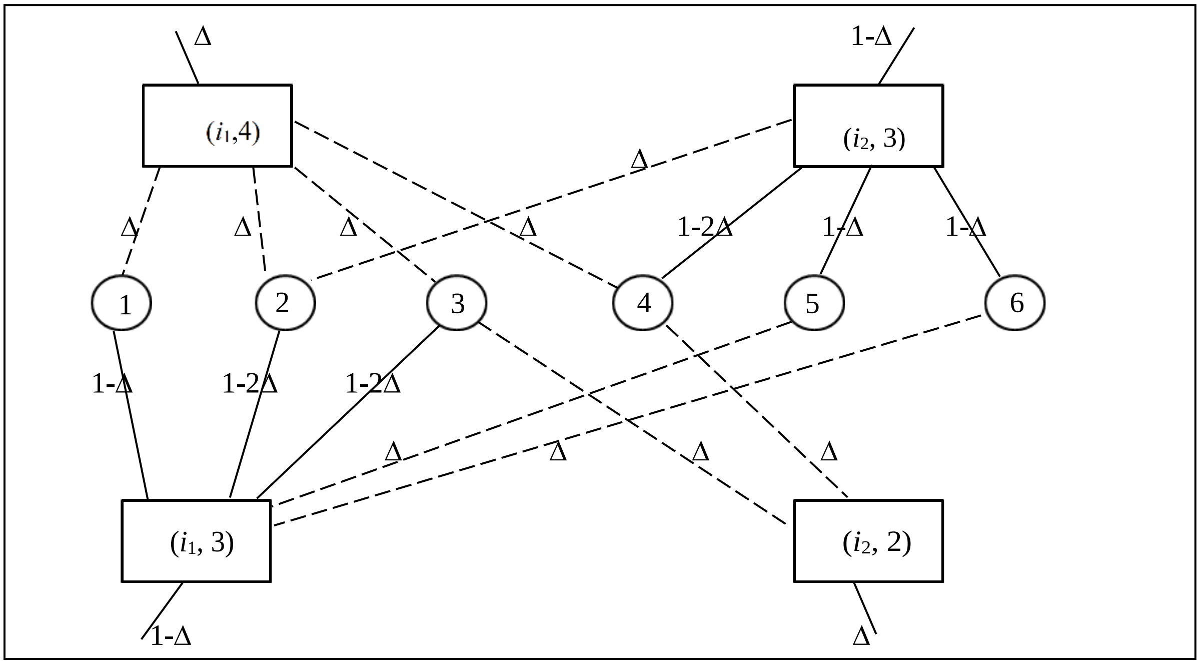

Figure 6: Illustration of movement from one integer solution to an adjacent non-integer solution.

6.0 Concluding Remarks and Future Research Possibilities

In this paper, we have presented a new disaggregated formulation of GAP that uses the idea of cardinality for each agent. The LP relaxation of this formulation is shown to be stronger than the LP relaxation of the standard formulation (**Ps**) used in earlier works such as Cattrysse et al. (1998) and Nauss (2003). The disaggregated formulation reveals generalizations of the well-known Cover and (1, *p*)-configuration inequalities that provide a much tighter description of the polytope, as well as being far more ubiquitous. Furthermore, this formulation reveals strong inequalities involving multiple agents. We present two such classes of inequalities: the Bar-and-Handle (1, $\hat{p}_k$) Inequality and the 2-Agent Cardinality Matching Inequality. Under certain restrictive conditions, the Bar-and-Handle (1, $\hat{p}_k$) inequality is shown to be a facet of the polytope defined by the feasible solutions of GAP. In the case of 2-Agent Cardinality Matching inequality, it first shown that it is a facet of the polytope defined by the feasible solutions to the un-capacitated version of GAP.

More importantly, it is shown that the lifting procedure needed to lift this inequality to become a facet of the polytope for the capacitated version is easy to solve. Finally, it is shown that for the special case of GAP consisting of just two agents, the 2-Agent Cardinality Matching inequality, along with trivial facets completely describe the polytope defined by the feasible solutions of the un-capacitated version of GAP. For the above reasons, we believe that these new inequalities, which are unique to the disaggregated formulation, are a significant contribution of this paper.

A disadvantage of the disaggregated formulation is its increased size, even though the increase is polynomially bounded. This can be a significant concern for large problems. One way to address it is to employ a *dynamic* reformulation that adds variables and constraints incrementally. Initially, we can start with a modified version of the traditional formulation (**Ps**) and then progressively move towards the disaggregated formulation. In the initial modified version of (**Ps**), variables $y_i \in \{0, 1\}$ for all $i \in M$ are introduced. Constraint (1) is replaced by the constraints:

$$\sum_{j \in N} a_{ij} x_{ij} \le b_i y_i \qquad \forall\ i \in M\ , \tag{49}$$

and the VUB constraints

$$x_{ij} \le y_i \qquad \forall\ i \in M, j \in N\ , \tag{50}$$

are introduced. After solving the LP relaxation, we measure $\sum_{j \in N} x_{ij}$ for each $i \in M$. Suppose that for some i, this aggregation is fractional and of value r_i. This agent is split into at most four cardinalities. The four cardinalities are: i) $k_{i1} = \lfloor r_i \rfloor - 1$, ii) $k_{i2} = \lfloor r_i \rfloor$, iii) $k_{i3} = \lfloor r_i \rfloor + 1$ and iv) $k_{i4} = \lfloor r_i \rfloor + 2$. The variable y_i is split into four variables wherein, $y_{ik_{i1}} + y_{ik_{i2}} + y_{ik_{i3}} + y_{ik_{i4}} = y_i$. Similarly, each variable x_{ij} is split into four such that $z_{ijk_{i1}} + z_{ijk_{i2}} + z_{ijk_{i3}} + z_{ijk_{i4}} = x_{ij}$. Accordingly, constraints (49) and (50) are also broken into four separate constraints for the agent that is split. The knapsack and the VUB constraints take the form:

$$\sum_{j \in N} a_{ij} z_{ij}^{k_{il}} \le b_i y_{ik_{il}} \qquad l = 1,2,3,4 \tag{51}$$

$$z_{ij}^{k_{il}} \le y_{ik_{il}} \qquad \forall\ j \in N, l = 1,2,3,4\ . \tag{52}$$

Finally, the following cardinality constraints are introduced for the agent that is split:

$$\sum_{j \in N} z_{ij}^{k_{i1}} \leq k_{i1} y_{ik_{i1}}, \tag{53}$$

$$\sum_{j \in N} z_{ij}^{k_{il}} = k_{il} y_{ik_{il}}, \qquad l = 2,3 \tag{54}$$

$$\sum_{j \in N} z_{ij}^{k4} \geq k_{i4} y_{ik_{i4}}. \tag{55}$$

With the introduction of cardinality constraints (53) and (54), the Cover inequalities as well as (1, $\hat{p}_k$)-Configuration inequalities discussed in Section 2 can now be applied on agents that have been split. The Bar-and-Handle inequalities and the 2-Agent Cardinality Matching inequalities can also be applied to agents that have been split. This form of judicious disaggregation can achieve the desired strength in the formulation without making the model unnecessarily large. Of course, the challenge lies in the implementation where rows and columns are added progressively, and this is a topic for future research.

An important future research issue from a computational standpoint is the separation problem associated with the various inequalities presented in this paper. As can be expected, the separation problems associated with these inequalities can be challenging. Another interesting idea worth exploring is to extend the 2-Agent Cardinality Matching inequalities to three agents. Once we have Cardinality Matching inequalities consisting of three agents, one can ‘concatenate’ sets of 2-Agent and 3-Agent Cardinality Matching inequalities to derive cardinality matching inequalities consisting of larger number of agents. Finally, similar disaggregation approaches can be investigated for other NP-Hard problems such as the Capacitated Concentrator Location Problem, or the Capacitated Network Design problem in which each commodity is either wholly assigned or not at all to each link in the network.

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