IIMB-WP N0. 720/2025

WORKING PAPER NO: 720

A new induction proof of the Gibbard-Satterthwaite theorem

Sarvesh Bandhu
Assistant Professor of Economics
Indian Institute of Management Bangalore
sarvesh.bandhu@iimb.ac.in

Rohit Kumar
Assistant Professor of Economics
Indian Institute of Technology, Delhi
rohitk@iitd.ac.in

Year of Publication – March 2025

A new induction proof of the Gibbard-Satterthwaite theorem*

Sarvesh Bandhu† and Rohit Kumar‡

Feburary 07, 2025

Abstract

In this paper, we provide a new induction proof of the Gibbard–Satterthwaite theorem, where the induction argument builds on $n = 1$ rather than the existing proofs using $n = 2$. The provided proof is much shorter, and the arguments are very intuitive, which will be helpful in teaching and for beginners in this field.

KEYWORDS. Social choice function, dictator, Gibbard–Satterthwaite theorem, induction.

JEL CODES D71, D82.

1 Introduction

The Gibbard-Satterthwaite theorem (henceforth, the GS Theorem) is a seminal work which showed a fundamental impossibility in social choice theory. Suppose a group of individuals is making a collective decision to select a winner among three or more alternatives. Each individual has a private (preference) ordering over the given set of alternatives. Is there a rule that will always induce individuals to submit orderings truthfully? The GS theorem states that the dictatorial rule is the only non-trivial rule that will incentivize individuals to report their private information truthfully.

There are many simple and elegant proofs of this result in the literature, e.g., Schmeidler & Sonnenschein (1978); Barberá (1983); Benoît (2000); Sen (2001); Reny (2001); Cato (2009); Ninjbat (2012); Yu (2013); Reffgen & Svensson (2014). One of the techniques among such proofs is the induction on the number of voters where induction starts with $n = 2$ (e.g. Sen (2001) and Reffgen & Svensson (2014)).[1] However, this paper observes that a direct proof using $n = 1$ is also possible, making the base case trivially true and significantly shortening the overall proof. To best our

*We thank Arunava Sen for suggesting this problem.

†Corresponding author, Department of Economics, Indian Institute of Management, Bangalore, India. Email: sarvesh.bandhu@iimb.ac.in.

‡Humanities & Social Sciences, Indian Institute of Technology, Delhi, India. Email: rohitk@iitd.ac.in.

[1] They first show theorem holds for two voters and then use induction to show that it holds for any number of voters.

knowledge, this is the first proof that starts the induction from $n = 1$. Our proof is elementary and suitable for new researchers and classroom teaching.

We want to highlight two features of our proof. Firstly, our proof is constructed in such a way that it works even over a restricted domain of free pair at the top, as in Sen (2001). For every pair a and b, this domain minimally contains two orderings such that a is first-ranked and b is second-ranked and vice-versa.[2] Finally, with minor modifications, we establish the GS theorem for another interesting restricted domain called the circular domain, introduced in Sato (2010).[3]

In the next section, we introduce our framework and the main definitions. Section 3 provides the statement and proof of the GS Theorem. Section 4 provides a proof for the circular domain, and Section 5 concludes.

2 Framework

Let $N = \{1, \ldots, n\}$ be a finite set of voters and A be a finite set of m alternatives i.e., $|A| = m$. Each voter $i \in N$ has a (linear) ordering P_i over the elements of the set A.[4] For distinct $a, b \in A$ by aP_ib we mean : a is strictly preferred to b by voter i according to her ordering P_i. Let $\mathcal{P}$ denote the set of all linear orderings over the elements of A. For any ordering P_i and integer $k = 1, \ldots, m$, $r_k(P_i)$ denotes the k^{th} ranked alternative in P_i, i.e. $|\{a \in A : aP_ir_k(P_i)\}| = k - 1$.

A profile is a list $P = (P_1, \ldots, P_n) \in \mathcal{P}^n$ of voters' orderings. For any coalition $S \subset N$, let $P_S \equiv (P_i)_{i \in S}$ and $P_{-S} \equiv (P_i)_{i \in N \setminus S}$. For simplicity, we write P_{-i} for $P_{-\{i\}}$ and P_{-ij} for $P_{-\{i,j\}}$ and so on. A profile P is also denoted by (P_i, P_{-i}), more generally (P_S, P_{-S}) for any $S \subset N$. Next we note the definitions and axioms used in the paper. They are very standard and widely known in the literature, so we skip the detailed explanations.

Definition **1.** *A social choice function (SCF) is a function or a mapping f such that $f : \mathcal{P}^n \to A$.*[5]

Definition **2.** *f is manipulable by voter i at profile P via P_i' if $f(P_i', P_{-i})\ P_i\ f(P)$. f is strategy-proof if it is not manipulable by any voter at any profile.*

Definition **3.** *f satisfies unanimity if for all $P \in \mathcal{P}^n$ and $a \in A$ such that $r_1(P_i) = a$ for all $i \in N$, we have $f(P) = a$.*

Definition **4.** *f is dictatorial if there exists a voter i (called a dictator) such that $f(P) = r_1(P_i)$ for all profiles P.*

[2]If there are m alternatives, then this domain requires only $m(m-1)$ orderings compared to the $m!$ orderings in the universal domain.

[3]This domain requires only $2m$ orderings. See Section 4 for more details.

[4]Linear order is a binary relation which satisfies completeness, transitivity, and anti-symmetry.

[5]As our proof works for a more general domain, so we could have defined SCF as $f : \mathcal{D}^n \to A$ where $\mathcal{D} \subset \mathcal{P}$. Now $\mathcal{D}$ could be free pair or circular domain.

3 Main result

THEOREM **1** (GS Theorem Gibbard (1973) and Satterthwaite (1975))**.** *Assume $|A| \geq 3$. Then a SCF f satisfies unanimity and strategy-proofness if and only if it is a dictatorial SCF.*[6]

Proof: **Step 1**. The theorem is (trivially) true for a single voter, $N = \{1\}$.
Step 2. Induction Hypothesis (IH): assume that for all integers $k < n$, if $f : \mathcal{P}^k \to A$ satisfies unanimity and strategy-proofness then it is a dictatorship. We will show that the above statement is true for $k = n$.

Define a SCF $g : \mathcal{P}^{n-1} \to A$ as follows: for all $(P_1, P_3, \ldots, P_n) \in \mathcal{P}^{n-1}$, $g(P_1, P_3, ..., P_n) = f(P_1, P_1, P_3, ..., P_n)$. In words, g is obtained from f by taking the ordering of voter 2 exactly same as voter 1. If f is strategy-proof and unanimous, so does the g. If all voters in g share the same top ranked alternative so does f and f being unanimous, it selects that alternative. So, g is also unanimous.

Now suppose at some profile (P_1, P_{-12}), g is manipulable by voter i via P'_i. If $i \geq 3$ then f is also manipulable by i at profile (P_1, P_1, P_{-12}) via P'_i. If $i = 1$ then f is manipulable by either 1 or 2. To see this let $g(P_1, P_{-12}) = a$ and $g(P'_1, P_{-12}) = b$ (obviously bP_1a). Let $f(P'_1, P_1, P_{-12}) = x$ (if $x = a$ then player 2 manipulates at (P'_1, P_1, P_{-12}) via P'_1). If xP_1a then voter 1 manipulates at (P_1, P_1, P_{-12}) via P'_1. So aP_1x. But then $f(P'_1, P'_1, P_{-12}) = g(P'_1, P_{-12}) = b$ and bP_1aP_1x. Hence, voter 2 manipulates at (P'_1, P_1, P_{-12}) via P'_1. Therefore, g must be strategy-proof.

Therefore, by applying IH, g is dictatorial. Let $i \in \{1, 3, \ldots, n\}$ be the dictator in g. We will consider two cases to establish that f is dictatorial.
Case 1: $i \in \{3, 4, \ldots, n\}$. We will show that i is also a dictator in f. Take an arbitrary profile P and assume $r_1(P_i) = a$. We need to show that $f(P) = a$. From IH, we have $f(P_1, P_1, P_{-12}) = a$ and $f(P_2, P_2, P_{-12}) = a$. Now suppose $f(P_1, P_2, P_{-12}) = b \neq a$. At ordering P_1, either we have bP_1a or aP_1b. Suppose it is bP_1a, then voter 2 can manipulate at (P_1, P_1, P_{-12}) via P_2. If aP_1b, then voter 1 can manipulate at (P_1, P_2, P_{-12}) via P_2. A contradiction of f being strategy-proof.
Case 2: $i = 1$. We will argue in steps that voter 1 or 2 is the dictator in f. We have also provided a visual description of each step in the Figure 1.
Step (i): Fix an arbitrary sub-profile $\bar{P}_{-12}$ and two (distinct) alternatives a and b. Consider a profile P such that $r_1(P_1) = a, r_2(P_1) = b, r_1(P_2) = b, r_2(P_2) = a$ and $P_{-12} = \bar{P}_{-12}$. Note that at profile P we have $f(P) = a$ or b, because if it is some $c \neq a, b$ then voter 1 can manipulate at P via P_2.[7] Let without loss of generality, $f(P) = a$. Now we will establish that voter 1 is the dictator in f.[8]
Step (ii): Consider any ordering P'_1 and P'_2 such that $r_1(P'_1) = a$ and $r_1(P'_2) = b$. Then $f(P'_2, P_{-2}) = a$ or b otherwise voter 1 can manipulate at (P'_2, P_{-2}) via P'_2. But $f(P'_2, P_{-2}) \neq b$ else voter 2 can manipulate at P via P'_2. Hence $f(P'_2, P_{-2}) = a$. Now, $f(P'_2, P_{-2}) = f(P'_1, P'_2, P_{-12}) = a$. If it is

[6]We could also write the theorem using onto-ness instead of unanimity. But they are equivalent under strategy-proofness.
[7]Or voter 2 can manipulate at P via P_1.
[8]Suppose if it was $f(P) = b$ then exactly similar arguments can establish that voter 2 is dictator in f.

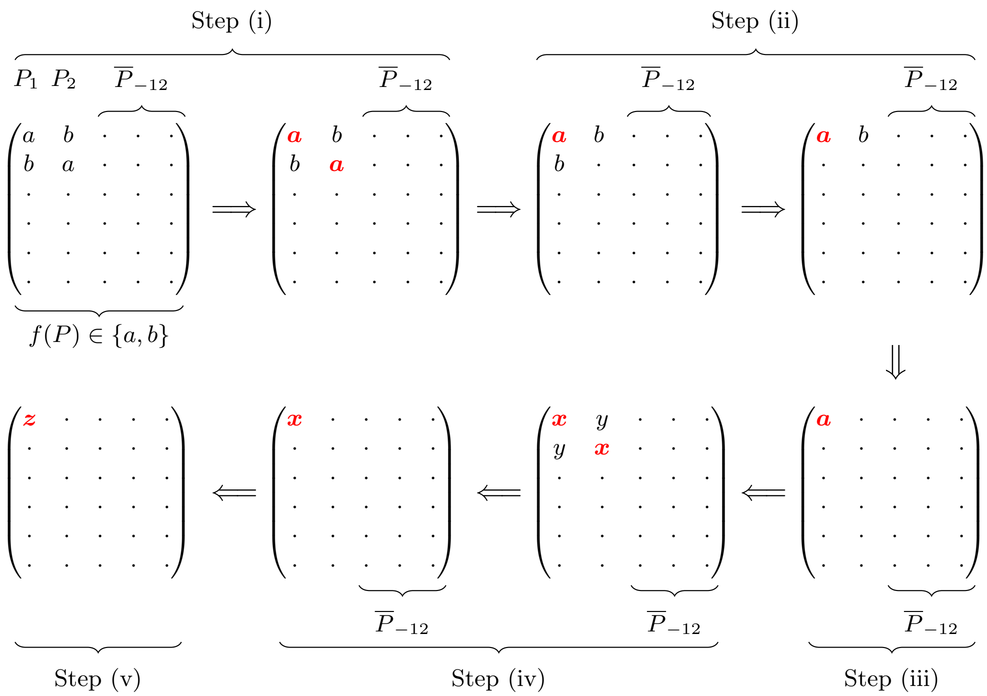

Figure 1: In each profile the outcome is shown by the bold red colored alternative. Note that every profile, except the last, is subjected to $\bar{P}_{-12}$.

not true then voter 1 can manipulate at (P_1', P_2', P_{-12}) via P_1.

Step (iii): For any ordering P_2'', we have $f(P_1', P_2'', P_{-12}) = a$. Suppose it is not true and we have $f(P_1', P_2'', P_{-12}) = c \neq a$. Now voter 2 can manipulate at (P_1', P_2', P_{-12}) via P_2'' when $cP_2'a$, which is true for an ordering where $r_1(P_2') = b$ and $r_2(P_2') = c$.[9] Hence we have established that voter 1 always gets alternative a when it is ranked top in her ordering.[10] Next, we show that outcome at all profiles is equal to her first ranked alternative i.e., voter 1 is indeed the dictator.[11]

Step (iv): Pick any two alternatives x and $y \neq a$. Consider a profile $\hat{P}$ where $r_1(\hat{P}_1) = x$, $r_2(\hat{P}_1) = y$, $r_1(\hat{P}_2) = y$, $r_2(\hat{P}_2) = x$ and $\hat{P}_{-12} = \bar{P}_{-12}$. Using the arguments of steps (i) to (iii), the outcome must be x or y whenever it is ranked first by voters 1 or 2, respectively. However, the latter is not possible as this contradicts Step (iii), where we have shown that voter 1 gets a whenever it is ranked first by her. Therefore, voter 1 always gets her top ranked alternative.[12]

Step (v): To establish that voter 1 is a dictator at all profiles, we need to show that choice of dictator between 1 and 2 does not depend on the sub-profile of voters from 3 to n. Recall the sub-profile $\bar{P}_{-12}$ from Step (i). Take any two alternatives, w and z such that $w\bar{P}_3 z$. Consider the profile P where $r_1(P_1) = z$, $r_2(P_1) = w$, $r_1(P_2) = w$, $r_2(P_2) = z$ and $P_{-12} = \bar{P}_{-12}$. According to Step

[9] Note that in this Step all we need to assume is the free pair at the top.

[10] This is also referred in the literature as "1 being decisive over a".

[11] Alternatively, 1 is decisive over all alternatives.

[12] To see this, if $f(\hat{P}) = y$, then first we can show that $f(\hat{P}_1', \hat{P}_{-1}) = y$, whenever $r_1(\hat{P}_1') = x$, then show that $f(\hat{P}_1', \hat{P}_2', \hat{P}_{-12}) = y$, whenever $r_1(\hat{P}_2') = y$ and finally for arbitrary $\hat{P}_1''$, $f(\hat{P}_1'', \hat{P}_2', \hat{P}_{-12}) = y$. This will contradict Step (iii) because it concludes that $f(\hat{P}_1'', \hat{P}_2', \hat{P}_{-12}) = a$ whenever $r_1(\hat{P}'') = a$.

(iii) we know that, $f(P) = z$. Now take an arbitrary ordering $\hat{P}_3$. Consider the profile, $(\hat{P}_3, P_{-3})$. According to Step (i), $f(\hat{P}_3, P_{-3}) = w$ or z. However, if it is w, then voter 3 can manipulate at P via $\hat{P}_3$. So it must be z. Now we can follow the arguments from Step (ii) to (iv) to establish that for arbitrary P'_1 and P'_2 we have $f(P'_1, P'_2, \hat{P}_3, \bar{P}_{-123}) = r_1(P'_1)$. We can apply the same arguments to arbitrary orderings from $\hat{P}_4$ through $\hat{P}_n$ to establish that voter 1 remains dictator. Because the choice of sub-profile $(\hat{P}_3, \hat{P}_4, \ldots, \hat{P}_n)$ is arbitrary, we have established that at every profile voter, 1 is the dictator.

The Case 1 and 2 complete the proof of Step 2. The Step 1 and 2 complete the proof of GS theorem.

4 Circular Domain

Sato (2010) introduced a domain called the circular domain.[13] We will show that our arguments are fairly general and can be adjusted to the circular domain with minor modifications. A different proof for this domain is almost identical as the previous proof. Below we *only* highlight the changes that are required in the steps of Case 2 of our proof.

Step (i'): Fix any two adjacent alternatives a and b.[14]

Step (iii'): For any arbitrary ordering P''_2 if $f(P'_1, P''_2, P_{-12}) = c \neq a$ then voter 2 can manipulate at (P'_1, P'_2, P_{-12}) via P''_2. This is possible whenever cP'_2a, which is true for an ordering where $r_1(P'_2) = b$ and $r_m(P'_2) = a$.[15]

Step (iv'): Pick any two alternatives x and y such that x and y are adjacent and $y \neq a$.

5 Conclusion

We provide an induction proof where the induction argument builds upon $n = 1$ rather than $n = 2$, as required in existing proofs. Our proof only requires the domain to satisfy the free pair at the top condition. No other restrictions are imposed on how outcomes are ranked lower down. In addition, we also provide a modified proof that works for the circular domain, which is even further smaller than the top pair domain in terms of number of possible orderings.

References

Barberá, S. (1983): "Strategy-Proofness and Pivotal Voters: A Direct Proof of the Gibbard-Satterthwaite Theorem," *International Economic Review*, 24(2) 413-417 .

[13] A domain of orderings is called circular domain if the alternatives can be arranged in a circle so that for every alternative x and for the alternatives adjacent to x on the circle, say, y and z, the following two conditions hold: (i) at some ordering in the domain, x is top-ranked, y is second-ranked and z is bottom ranked, and (ii) at some other ordering x is top ranked, z is second ranked, and y is bottom ranked. For further discussion, see Sato (2010).

[14] For the definition of adjacent alternatives see Sato (2010).

[15] Such an ordering exits because in Step (i') we have selected two adjacent alternatives – a and b.

Benoît, J. P. (2000): "The Gibbard–Satterthwaite theorem: a simple proof," *Economics Letters*, 69(3), 319-322.

Cato, U. (2009): "Another induction proof of the Gibbard–Satterthwaite theorem," *Economics Letters*,105(3), 239-241.

Gibbard, A. (1973): "Manipulation of voting schemes: a general result," *Econometrica*, 41(4), 587–601.

Ninjbat, U. (2012): "Another direct proof of the Gibbard–Satterthwaite theorem," *Economics Letters*, 116(3), 418-421.

Reffgen, A., and L.G. Svensson (2014): "The proof of the Gibbard–Satterthwaite theorem revisited," *Journal of Mathematical Economics*, 10(2), 187–217.

Reny, P.J. (2001): "Arrow's theorem and the Gibbard-Satterthwaite theorem: a unified approach," *Economics Letters*, 70(1), 99-105.

Sato, S. (2010): "Circular domains", *Review of Economic Design*, 14(3), 331–342.

Satterthwaite, M. (1975): "Strategy-proofness and Arrow's conditions: Existence and correspondence theorems for voting procedures and social welfare functions," *Journal of economic theory*, 10(2), 187–217.

Schmeidler, D., and H. Sonnenschein (1978): "Two Proofs of the Gibbard-Satterthwaite Theorem on the Possibility of a Strategy-Proof Social Choice Function," *Decision Theory and Social Ethics*, chapter(175), 227–234.

Sen, A. (2001): "Another direct proof of the Gibbard–Satterthwaite theorem," *Economics Letters*, 70(3), 381-385.

Yu, N.N. (2013): "A one-shot proof of Arrow's theorem and the Gibbard–Satterthwaite theorem", *Economic Theory Bulletin*, 1, 145–149.