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Robust Procurement Design

Debasis Mishra
Professor of Economics
Indian Statistical Institute, Delhi
dmishra@isid.ac.in

Sanket Patil
Assistant Professor of Economics
Indian Institute of Management Bangalore
sanket.patil@iimb.ac.in

Alessandro Pavan
Professor of Economics
Northwestern University
alepavan@northwestern.edu

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Debasis Mishra, Sanket Patil, and Alessandro Pavan†

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Abstract

We study procurement design when the buyer is uncertain about both the value of the good and the seller's cost. The buyer has a conjectured model but does not fully trust it. She first identifies mechanisms that maximize her worst-case payoff over a set of plausible models, and then selects one from this set that maximizes her expected payoff under the conjectured model. Robustness leads the buyer to increase procurement from the least efficient sellers and reduce it from those with intermediate costs. We also study monopoly regulation and identify conditions under which quantity regulation outperforms price regulation.

KEYWORDS: Robust mechanism design, uncertainty, procurement, regulation.

JEL CLASSIFICATION: D82, L51

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†Mishra: Indian Statistical Institute, Delhi, dmishra@isid.ac.in. Patil: Indian Institute of Management Bangalore, sanket.patil@iimb.ac.in. Pavan: Northwestern University, alepavan@northwestern.edu.

1 Introduction

Hoping for the best, prepared for the worst, and unsurprised by anything in between.

– Maya Angelou, *I Know Why the Caged Bird Sings*

Procurement plays a central role in economics. Governments rely on sophisticated contracts to purchase goods and services for citizens and to regulate firms with market power. Similarly, consumers and firms use customized contracts in bilateral trades for major assets, ranging from real estate to inputs used in production. A key difficulty in procurement design is that sellers typically hold private information about their costs, which forces the buyer to leave rents to induce truthful revelation. A second challenge is that both the value of the good and the seller's cost technology may be uncertain at the contracting stage. The standard approach assumes the buyer has a "conjectured model" specifying the value for the good (i.e., a demand curve) and a cost technology (i.e., a distribution from which the seller's cost is drawn), with both obtained from the estimation, or calibration, of past data, or by integrating over a set of alternative models using a subjective belief. The buyer then selects the mechanism that maximizes her expected payoff (under the conjectured model) subject to incentive and participation constraints. This approach is often referred to in the literature as "Bayesian".

In many settings, however, the designer may not fully trust her model. The lack of trust may come, for example, from the fear that the data used to estimate or calibrate the conjectured model may no longer be representative of the relevant environment. This paper studies optimal design when the designer seeks robustness to such model misspecification. For concreteness, we focus on a government buyer in the introduction, though the results apply equally to private procurement and other design problems. The buyer first identifies all *worst-case optimal mechanisms*—those maximizing her minimum payoff across a set of plausible models (with each model comprising (a) a demand function specifying the social, or private, gross value of the quantity procured, and (b) a cost technology specifying the distribution from which the seller's cost is drawn). The buyer then selects, among them, the mechanism that maximizes her expected payoff, under the conjectured model. We refer to a mechanism solving this design problem as *robustly optimal.* We assume the designer's objective is to maximize consumer surplus. This assumption allows the results to carry over directly to private procurement and, after role reversal, to seller-side screening problems such as Mussa and Rosen (1978).

This two-step approach can be interpreted as arising from organizational frictions—for example, the need to obtain approval from a supervisor requiring that the adopted mechanism deliver a satisfactory payoff guarantee. For instance, in the oil and gas sector, senior management commonly requires that projects remain viable under conservative price scenarios—often corresponding to oil prices well below prevailing market conditions—before approving investment. In the main text, we assume this guarantee is the maximal attainable one (given the set of plausible models, and the need to provide incentives to the seller). In the supplement, we consider more permissive short lists comprising all mechanisms for which the payoff guarantee is no less than a required fraction of the maximal attainable one.

Our first result concerns settings where uncertainty is only over the seller's cost technology, and the demand function (equivalently, the value of the good) is known to the designer. The buyer considers a set of possible cost distributions, each representing a distinct technology.[1] When the conjectured distribution satisfies standard regularity conditions, the optimal mechanism takes the form of *Baron–Myerson with a quantity floor*. In the classical Baron and Myerson (1982) model, each cost type of the seller produces a quantity distorted downward relative to the efficient level, with no distortion for the lowest-cost type. Under robustness, the buyer retains the same schedule but imposes a floor ensuring every type supplies at least the efficient output of the highest-cost type. The floor protects against the possibility that high-cost firms occur more frequently than conjectured. Because downward distortions serve primarily to limit rents for low-cost types, their value diminishes under uncertainty. Hence, robustness calls for higher quantities from high-cost types of sellers—achieving efficiency at both ends of the cost distribution. This mechanism differs from other worst-case-optimal ones that are not robustly optimal under our criterion. For example, a single contract where quantity is equal to the efficient level for the highest-cost type and the transfer is equal to the cost of for such a type to supply such a quantity is worst-case optimal but dominated by the Baron–Myerson-with-quantity-floor mechanism: It delivers a lower payoff to the designer for all technologies, strictly for a subset of the relevant range.

The combination of a quantity floor (designed for a range of high-cost firms) with a menu of quantity levels for lower-cost sellers is a feature of many procurement contracts.[2] Our anal-

[1]In the main text, this set comprises all distributions with support contained in a given interval. In the online supplement, we allow for more general sets.

[2]For example, in defense procurement, indefinite-delivery/indefinite-quantity (IDIQ) contracts with cost-

ysis suggests this combination may be a natural response to model uncertainty, independently of concerns about cost overruns or moral hazard. The optimality of the Baron–Myerson-with-quantity-floor mechanism extends to some environments where the government also faces uncertainty over the demand of the good—for instance, when the conjectured demand is the smallest within the set of plausible demands. More generally, we provide necessary and sufficient conditions for the Baron–Myerson-with-quantity-floor mechanism to remain robustly optimal under demand uncertainty and characterize the qualitative features of robustly optimal mechanisms when it is not. In such cases, robustness requires the same quantity for low-cost and high-cost types as in the Baron–Myerson-with-quantity-floor mechanism, but smaller quantities for intermediate-cost types. The intuition is the following. At the top of the cost distribution, robustness requires procuring the efficient output under the lowest demand and highest cost. Optimality (under the conjectured model) then requires procuring the same output also from an entire interval of costs around the highest level. For intermediate types, instead, where the floor does not bind, downward adjustments are needed to protect against the possibility that the actual demand is lower than expected. By contrast, for low costs, the buyer's payoff exceeds the guarantee irrespective of the realized demand, making any adjustment (vis-a-vis the Bayesian optimum) unnecessary.

We then study environments with a downstream market, where demand can be discovered by letting firms sell directly to consumers. We focus on price mechanisms, in which the government sets a price for each type of seller and lets the firm meet consumer demand at that price. Such mechanisms naturally hedge against demand misspecification by making procurement responsive to realized demand but expose firms to demand uncertainty. Here too we adopt a robust approach by requiring the price mechanism to be individually rational and incentive compatible no matter the firm's beliefs over the demand (formally, this is accomplished by conditioning the transfer to the firm on the realized demand). We show that robustly optimal price mechanisms are simple: each type is asked to apply the Bayesian optimal markup (under the designer's model), with a cap binding for high-cost types. The cap protects against the possibility that high-cost firms are more prevalent than expected.

Finally, we compare price and quantity regulation. Under Bayesian analysis, price reg-

plus-incentive-fee (CPIF) payment structures specify a minimum order or minimum guaranteed dollar value, while allowing incentive fees to be conditioned on delivery and performance milestones (Federal Acquisition Regulation 16.504(a) and 16.405-1). Similar minimum purchase guarantees are also used in government medical procurement, including vaccines and medical countermeasures.

ulation dominates quantity regulation because it implements the optimal quantity schedule for each possible demand. Under robust optimality, this ranking no longer holds. While both types of regulations offer the same maximal guarantee, their expected payoffs under the conjectured model differ. When the Baron–Myerson-with-quantity-floor mechanism is robustly optimal, quantity regulation dominates—strictly so when the quantity demanded at the highest marginal cost under the conjectured model exceeds the one under the lowest demand in the admissible set. This is because the extent of over-procurement from high-cost firms is larger under price regulation. Conversely, when the quantity demanded under the conjectured model at the highest cost coincides with the quantity demanded under the lowest demand, price regulation dominates, as it avoids the downward adjustment for intermediate-cost types required under quantity regulation.

These findings yield primitive conditions determining which form of regulation dominates, depending on whether uncertainty over demand is concentrated at low or high prices and whether the estimated cost distribution puts more weight on low or high costs.

Organization. The remainder of the paper is organized as follows. We conclude the introduction with a brief discussion of the relevant literature. Section 2 presents the environment and the buyer's problem of designing a robustly optimal procurement mechanism. Section 3 characterizes the set of worst-case-optimal mechanisms. Section 4 derives the properties of robustly optimal mechanisms. Section 5 extends the analysis to settings with a downstream market, characterizes robustly optimal price mechanisms, and identifies conditions under which quantity regulation dominates (or is dominated by) price regulation. Section 6 concludes. All proofs not in the main text are in Appendices A–D. The article's online supplementary material, Mishra et al. (2025), contains additional results: (a) it establishes existence of robustly optimal mechanisms, (b) shows that robustly optimal mechanisms are undominated, (c) discusses how the results extend to a more permissive short list containing mechanisms for which the guarantee is no less than a fraction $\gamma \in [0, 1]$ of the maximal one, (d) considers more general forms of cost uncertainty in which the most adversarial cost distribution is less adversarial than in the main text, and (e) shows that the output procured under robustly optimal mechanisms need not be monotone in the uncertainty the designer faces over the set of feasible cost technologies.

Related literature. A vast literature in information economics studies optimal mechanisms when agents hold private information about key primitives—preferences, costs, technology, or productivity. The closest works to ours in the procurement and regulation domains are Baron and Myerson (1982) and Laffont and Tirole (1986). Related analyses include Armstrong (1999), Amador and Bagwell (2022), Armstrong and Sappington (2006), Biglaiser and Ma (1995), Dana (1993), Lewis and Sappington (1988), and Yang and Zentefis (2023). These papers explore how optimal mechanisms depend on the designer's information about demand and cost, under the assumption that the designer fully trusts a specified model—what we refer to as the Bayesian benchmark.[3]

Recent work has relaxed the key assumptions underlying the Bayesian analysis to develop robust approaches to contract design. Carroll (2019) provides an excellent survey of this literature. The most closely related papers are Garrett (2014), Bergemann et al. (2023), Guo and Shmaya (2025), and Kambhampati (2025). Garrett (2014) characterizes optimal contracts when the designer lacks information about a manager's disutility from effort in Laffont and Tirole (1986) setting. Bergemann et al. (2023) study robustness based on worst-case competitive ratios. Guo and Shmaya (2025) analyze min–max regret and identify conditions for the optimality of price-cap regulation under private information about both demand and costs.[4] Kambhampati (2025) considers lexicographic worst-case optimality when mechanism performance is evaluated across a system of multiple beliefs. Our analysis is also lexicographic but differs in that the designer's second-order belief is given by the conjectured model, which is a primitive of the environment. This distinction has important implications for the structure of the optimal mechanism. For example, we show that the tension between efficiency and rent extraction persists in the presence of robustness concerns. None of the above papers identifies conditions under which quantity regulation outperforms price regulation, which is one of the distinctive contributions of our work. Related are also Börgers et al. (2025) and Mishra and Patil (2025) who consider the design of undominated mechanisms, respectively in auctions and regulation. In all these papers, the designer does not have a conjectured model and treats all models in the admissible set equally. In contrast, in our approach, the designer does have a conjectured model and uses it to select among mechanisms that yield the largest payoff guarantee. This feature aligns our analysis with Dworczak and Pavan

[3]The properties of Bayesian optimal mechanisms for a monopolist selling to consumers with private information about their willingness to pay are qualitatively identical to those in the procurement model, modulo a reversal in the players' roles – see, e.g., Mussa and Rosen (1978).

[4]See also Bergemann and Schlag (2008) for an earlier analysis of min-max regret in monopoly pricing and Segal (2003) for multi-unit auction design when the demand is unknown.

(2022), who study robust information design without transfers or screening. In contrast, we consider mechanism design with transfers and private information on the agent's side.

Our paper is also related to the literature on model misspecification and robust control (see Cerreia-Vioglio et al. (2025) for an overview). In that literature, performance under alternative models is weighted by a distance measure between each model and the conjectured one. In contrast, our designer treats all models within the admissible set symmetrically when computing the payoff guarantee, but then uses the conjectured model to select from the short list of mechanisms for which the payoff guarantee is the highest. In this respect, our approach also differs from Madarász and Prat (2017) who consider model misspecification in a monopolistic screening setting in which non-local incentive constraints bind and show that small misspecification can lead to large losses.

Finally, in settings with downstream markets, our analysis connects—conceptually—with Valenzuela-Stookey and Poggi (2025), who study the use of markets as instruments for implementing desired allocations. See also Weitzman (1974) for an early analysis of price vs quantity regulation in a setting without private information where the regulator does not seek robustness.

None of the existing studies generate the predictions about the structure of robustly optimal mechanisms and the comparison of price and quantity regulation that emerge from our analysis.

2 Model

2.1 Environment

A buyer (a government agency or a private organization) procures a product or service. The good is supplied by a monopolistic seller who can provide any quantity $\mathrm{q} \in [0, \bar{\mathrm{q}}]$, where $\bar{\mathrm{q}} \in \mathbb{R}_{++}$ is finite but large enough that it is never optimal for the buyer to procure more than $\bar{\mathrm{q}}$, regardless of the buyer's valuation or the seller's cost (the precise condition ensuring this property is provided below).[5]

[5]The assumption that $\bar{\mathrm{q}}$ is finite guarantees that the seller's equilibrium payoff satisfies a familiar envelope-theorem representation.

The seller's cost of supplying $q \geq 0$ is θq, where the marginal cost θ is the seller's private information.[6]

The buyer is uncertain about the gross value of procuring q units. Based on past data, the buyer's conjecture is that this value is given by an increasing, concave, and differentiable function $V^\star : \mathbb{R}_+ \to \mathbb{R}_+$, with derivative $P^\star$ (interpreted as the inverse demand function) such that, for any q, $V^\star(q) = \int_0^q P^\star(s)ds$.[7] However, the buyer is concerned that the true value function may differ from $V^\star$. Specifically, the buyer considers a set $\mathcal{V}$ of admissible value functions, each increasing, concave, and differentiable, with associated set of inverse demand functions $\mathcal{P}$. That is, for each $V \in \mathcal{V}$, there is an inverse demand function $P \in \mathcal{P}$ representing the derivative of V such that

$$V(q) = \int_0^q P(s)ds \tag{1}$$

for all $q \geq 0$. Representing each V as the integral of its inverse demand function P facilitates the application to monopoly regulation in Section 5.1. For each $P \in \mathcal{P}$, let D denote the corresponding direct demand, defined by $D(p) = P^{-1}(p)$ for all $p \in [0, P(0)]$ and $D(p) = 0$ for all $p > P(0)$. Let $\mathcal{D}$ denote the set of all such direct demand functions and $D^\star$ the demand associated with $P^\star$. The assumption that $\bar{q}$ is large is then equivalent to the property that $D(\underline{\theta}) < \bar{q}$ for all $D \in \mathcal{D}$.

The sets $\mathcal{V}$, $\mathcal{P}$, and $\mathcal{D}$ are such that $(V^\star, P^\star, D^\star) \in \mathcal{V} \times \mathcal{P} \times \mathcal{D}$. We assume that there exists a "smallest" inverse demand $\underline{P} \in \mathcal{P}$ such that $P(q) \geq \underline{P}(q)$ for all $q \geq 0$ and all $P \in \mathcal{P}$. The function $\underline{P}$ provides a lower bound on marginal value and is decreasing and continuous. The smallest inverse demand function $\underline{P}$ induces the smallest value function $\underline{V}$.

The buyer is also uncertain about the cost technology, captured by the distribution from which the seller's marginal cost θ is drawn. The buyer's conjecture is that θ is drawn from a regular distribution $F^\star$ with support $\Theta = [\underline{\theta}, \overline{\theta}] \subset \mathbb{R}_{++}$; that is, from a cdf $F^\star$ that is absolutely continuous over $\mathbb{R}$ with density $f^\star$ strictly positive over Θ and such that $F^\star(\underline{\theta}) = 0$, $F^\star(\overline{\theta}) = 1$, and the virtual cost $z(\theta) \equiv \theta + F^\star(\theta)/f^\star(\theta)$ is continuous and

[6]All our results extend to a setting with a fixed cost $c \in (0, \bar{c})$, where $\bar{c} \in \mathbb{R}_{++}$ is a bound whose role is to guarantee that it is optimal for the buyer to procure a positive quantity even when expecting the largest marginal cost and the lowest value for the good.

[7]Throughout, a function g is said to be increasing (alternatively, weakly increasing) if $g(x) > g(x')$ (alternatively, $g(x) \geq g(x')$) whenever $x > x'$. The terms *decreasing* and *weakly decreasing* are analogously defined. Similarly, when we say that a function is concave, convex, or quasi-concave, we mean strictly.

increasing over Θ. The buyer considers a set $\mathcal{F}$ of admissible cost distributions. We assume that $\mathcal{F} = \text{CDF}(\Theta)$, where $\text{CDF}(\Theta)$ is the set of all cdfs whose support is contained in Θ; that is, each $F \in \text{CDF}(\Theta)$ is a weakly increasing, right-continuous function $F : \mathbb{R} \to [0, 1]$ such that $F(\theta) = 0$ for all $\theta < \underline{\theta}$, and $F(\theta) = 1$ for all $\theta \geq \overline{\theta}$. The supplement considers the case in which $\mathcal{F}$ does not coincide with $\text{CDF}(\Theta)$ and discusses how the results depend on the lowest element of such a set (in the main text, the lowest element is a Dirac assigning probability one to $\overline{\theta}$; in the supplement, we allow the lowest element of $\mathcal{F}$ to be a non-degenerate cdf).

We assume that $\overline{\theta} < \lim_{\mathrm{q}\downarrow 0} \underline{P}(\mathrm{q})$. Together with the continuity of $\underline{P}$, this assumption ensures that there are gains from trade no matter the seller's cost and the inverse demand.

Lastly, we assume that, once the buyer learns V, ex-post adjustments to output are either infeasible or not worthwhile. Such adjustments may be prohibitively costly, or the marginal value of additional output may be too low to justify the extra production cost. In the absence of these frictions, uncertainty over V is inconsequential.

In summary, $(V^\star, F^\star)$ represents the buyer's conjectured model, with $V^\star$ capturing the value of procuring output (with $P^\star$ denoting its marginal value and $D^\star$ the conjectured, or estimated, demand) and $F^\star$ capturing the cost technology. The buyer fears that the true model may be some alternative $(V, F) \in \mathcal{V} \times \mathcal{F}$, where $\mathcal{V} \times \mathcal{F}$ is the relevant admissible set.

2.2 Procurement mechanisms

To elicit the seller's private information and discipline procurement, the buyer offers a (direct) mechanism $M = (q, t)$. The quantity schedule $q : \Theta \to [0, \bar{\mathrm{q}}]$ specifies the procured quantity as a function of the reported cost, whereas the transfer schedule $t : \Theta \to \mathbb{R}$ specifies the total payment.[8]

The mechanism $M = (q, t)$ is *incentive compatible* (IC) if, for all $\theta, \theta' \in \Theta$,

$$u(\theta) \equiv t(\theta) - \theta q(\theta) \geq t(\theta') - \theta q(\theta') = u(\theta') + (\theta' - \theta) q(\theta').$$

[8]The focus on deterministic mechanisms is without loss in our setting because, for any given θ, the buyer's welfare is concave in q, and because Nature chooses (V, F) in response to M, in which case the buyer cannot hedge by randomizing over the elements of the mechanism. As it will become clear from the analysis below, the mechanisms that solve the designer's problem remain optimal even if the designer expects Nature to choose (V, F) simultaneously with her choosing M. This is because the solution to the designer's problem is a saddle point.

It is *individually rational* (IR) if $u(\theta) \geq 0$ for all $\theta \in \Theta$. Because, given the quantity schedule q, there is a bijection between the transfer schedule t and the utility/rent schedule u, we will often refer to a mechanism by (q, u) instead of (q, t).

As is standard, $M = (q, u)$ is IC and IR if and only if (a) q is weakly decreasing, and (b) for all $\theta \in \Theta$, $u(\theta) = u(\overline{\theta}) + \int_{\theta}^{\overline{\theta}} q(y)dy$, with $u(\overline{\theta}) \geq 0$.

Let $\mathcal{M}$ be the set of all IC and IR mechanisms. If the buyer's gross value is $V \in \mathcal{V}$ and the technology is $F \in \mathcal{F}$, the buyer's ex-ante welfare under the mechanism $M \equiv (q, u) \in \mathcal{M}$ is

$$W(M; V, F) \equiv \int \left(V(q(\theta)) - \theta q(\theta) - u(\theta)\right) F(\mathrm{d}\theta),$$

where $V(q(\theta)) - \theta q(\theta) - u(\theta) = V(q(\theta)) - t(\theta)$ is the buyer's net value when the cost is θ.

2.3 Buyer's problem

The buyer selects a mechanism through a two-step procedure.

Step 1 (guarantee maximization). Each IC and IR mechanism $M \in \mathcal{M}$ is evaluated by its **welfare guarantee** defined by

$$G(M) \equiv \inf_{V \in \mathcal{V}, F \in \mathcal{F}} W(M; V, F).$$

The buyer then computes a **short list** of mechanisms comprising those with the highest guarantee:

$$\mathcal{M}^{\mathrm{SL}} \equiv \arg\max_{M \in \mathcal{M}} G(M).$$

Any mechanism $M \in \mathcal{M}$ is thus **worst-case optimal**. A more permissive definition of short list is the collection of all mechanisms for which the guarantee is no smaller than a fraction $\gamma \in [0, 1]$ of the maximal attainable guarantee (formally, $\mathcal{M}^{\mathrm{SL}}(\gamma) \equiv \{M \in \mathcal{M} : G(M) \geq \gamma G(M') \ \forall \ M' \in \mathcal{M}\}$. We discuss this possibility in the online supplement.[9]

[9] See also Andrews and Chen (2025) for a different problem but with a similar short list. That paper considers a decision-maker (DM) choosing between a default action and an analyst-recommended action based on data unobservable to the DM. The DM selects the recommended action if the following two conditions are met: (1) the worst-case payoff of the recommended action is not too far below that of the default action; and (2) the expected payoff of the recommended action under the DM's model exceeds the expected payoff of the default action, again under the DM's model.

Step 2 (selection under conjectured model). Among mechanisms in the short list, the buyer selects the one that maximizes expected welfare under the conjectured model $(V^\star, F^\star)$.

Definition 1 *A mechanism M is* ***robustly optimal*** *if*

$$M \in \arg\max_{M' \in \mathcal{M}^{\mathrm{SL}}} W(M'; V^*, F^*).$$

3 Short-list characterization

We begin by deriving the maximal guarantee of an arbitrary IC and IR mechanism and show that the worst-case welfare need not occur under the technology that places all probability mass at $\overline{\theta}$, but always arises under the lowest possible demand function $\underline{D}$.

Let

$$\mathrm{q}_\ell \equiv \arg\max_{\mathrm{q}\in[0,\bar{\mathrm{q}}]} \left\{\underline{V}(\mathrm{q}) - \overline{\theta}\mathrm{q}\right\} = \underline{D}(\overline{\theta})$$

be the unique quantity that maximizes total surplus when $V = \underline{V}$ and $\theta = \overline{\theta}$; i.e., q_ℓ is the efficient quantity at the lowest demand and highest cost. Define

$$G^* \equiv \underline{V}(\mathrm{q}_\ell) - \overline{\theta}\mathrm{q}_\ell,$$

the welfare when gross value is lowest, cost is highest, and the buyer procures q_ℓ.

Lemma 1 (guarantee) *For any IC and IR mechanism $M = (q, u) \in \mathcal{M}$,*

$$G(M) = \inf_{\theta\in\Theta} \{\underline{V}(q(\theta)) - \theta q(\theta) - u(\theta)\} \tag{2}$$

and

$$G(M) \leq G^*. \tag{3}$$

The first part of Lemma 1 highlights that Nature can reduce the buyer's welfare more effectively by selecting a cost $\theta < \overline{\theta}$. Since q is weakly decreasing, the buyer procures more from lower-cost types; thus, when Nature selects an inverse demand below $P^\star$, the welfare loss from over-procurement can be larger at low θ.

The second part states that the welfare guarantee of any IC and IR mechanism cannot exceed the total surplus obtained from procuring the efficient output q_ℓ when demand is lowest and cost is highest. This follows because Nature can always choose the lowest demand and a degenerate cost distribution at $\overline{\theta}$, in which case the buyer's best response is to procure q_ℓ.

The next lemma shows that the upper bound on the maximal welfare guarantee is tight and fully characterizes the short list $\mathcal{M}^{\mathrm{SL}}$.

Lemma 2 (short-list characterization) *Take any IC and IR mechanism $M \equiv (q, u) \in \mathcal{M}$. Then, $M \in \mathcal{M}^{\mathrm{SL}}$ if and only if (a) $u(\overline{\theta}) = 0$, and (b), for all $\theta \in \Theta$,*

$$\underline{V}(q(\theta)) - \theta q(\theta) - \int_{\theta}^{\overline{\theta}} q(y)dy \geq G^{*}. \tag{4}$$

Worst-case optimality therefore imposes two additional constraints beyond IC and IR. First, the highest-cost type $\overline{\theta}$ must earn zero rent: otherwise, Nature can select $\underline{V}$ and a degenerate cost distribution at $\overline{\theta}$, reducing welfare strictly below $G^{\star}$ regardless of $q(\overline{\theta})$. Consequently, for any $M \in \mathcal{M}^{\mathrm{SL}}$, the rent schedule u is pinned down by the output schedule q. Second, ex-post welfare under the lowest possible gross value function and zero rent for $\overline{\theta}$ must be weakly above the maximal guarantee $G^{\star}$, *for any cost θ*. Sufficiency follows from Lemma 1; necessity is shown in Appendix A by constructing a simple constant mechanism that attains $G^{\star}$, namely one that procures q_ℓ from all types. This constant mechanism, however, is only one element of a continuum in the short list. Indeed, Condition (4) is equivalent to the requirement that (see Lemma 4 in Appendix D)

$$\int_{\theta}^{\overline{\theta}} q(y)dy \leq \int_{\theta}^{\overline{\theta}} \underline{D}(y)dy - \underbrace{\int_{\theta}^{\underline{P}(q(\theta))} \left(\underline{D}(y) - q(\theta)\right) dy}_{\underline{\mathrm{DWL}}(\theta, q(\theta)) \geq 0}, \tag{5}$$

where the last term represents the deadweight loss incurred under $\underline{V}$ when output $q(\theta)$ is procured instead of the efficient quantity $\underline{D}(\theta)$. Clearly, the efficient schedule $q = \underline{D}$ satisfies the robustness constraint in (5). More generally, the following holds:

Observation 1 (quantity bound) *If $M \equiv (q, u) \in \mathcal{M}^{\mathrm{SL}}$, then $q(\theta) \geq \mathrm{q}_\ell$ for all $\theta \in \Theta$, with equality at $\overline{\theta}$.*

Proof: Since $M = (q, u)$ belongs to $\mathcal{M}^{\text{SL}}$, by Lemma 2, the robustness constraint (4) must hold at $\overline{\theta}$:

$$\underline{V}(q(\overline{\theta})) - \overline{\theta}q(\overline{\theta}) \geq G^* = \underline{V}(\text{q}_\ell) - \overline{\theta}\text{q}_\ell \geq \underline{V}(q(\overline{\theta})) - \overline{\theta}q(\overline{\theta}).$$

Hence, $q(\overline{\theta}) = \text{q}_\ell$. Since q is weakly decreasing, $q(\theta) \geq q(\overline{\theta}) = \text{q}_\ell$ for all $\theta \in \Theta$. ■

Thus, any mechanism $M \equiv (q, u) \in \mathcal{M}^{\text{SL}}$ procures no less than q_ℓ from any type. For any weakly decreasing schedule q that procures $q(\theta) \in [\text{q}_\ell, \underline{D}(\theta)]$ from each type θ, the robustness constraint in (5) reduces to

$$\int_{\theta}^{\overline{\theta}} q(y)dy \leq \int_{\theta}^{\overline{\theta}} \min\{\underline{D}(y), q(\theta)\}dy,$$

which is always satisfied, and, consequently, such a q corresponds to a mechanism in the short list. Moreover, $\mathcal{M}^{\text{SL}}$ also includes mechanisms that procure more than the efficient output $\underline{D}(\theta)$ for some types—a feature that plays a key role in the analysis below.

4 Robustly optimal mechanisms

Using Lemma 2 and the standard "virtual surplus" representation of total welfare under the conjectured model $(V^\star, F^\star)$ (see Baron and Myerson (1982)), the buyer's problem can be written as follows. Recall the definition of the virtual cost function

$$z^\star(\theta) \equiv \theta + \frac{F^\star(\theta)}{f^\star(\theta)} \qquad \forall\, \theta \in \Theta.$$

Because $F^\star$ is regular, $z^\star$ is increasing. The robustly optimal quantity schedule then solves

$$\max_q \int_{\underline{\theta}}^{\overline{\theta}} \Big[V^\star(q(\theta)) - z^\star(\theta)q(\theta)\Big] F^\star(\text{d}\theta) \qquad \textbf{(ROPT)}$$

subject to q weakly decreasing, and

$$\underline{V}(q(\theta)) - \theta q(\theta) - \int_{\theta}^{\overline{\theta}} q(y)dy \geq G^* \qquad \forall\, \theta \in \Theta.$$

Relative to Baron and Myerson (1982), this program introduces an additional robustness constraint requiring that, for each $\theta \in \Theta$, ex-post welfare under the lowest possible demand exceeds the guarantee G^*.

We denote a quantity schedule solving (**ROPT**) as q^{OPT} and then let $M^{\mathrm{OPT}} \equiv (q^{\mathrm{OPT}}, u^{\mathrm{OPT}})$ be the mechanism corresponding to q^{OPT}, with u^{OPT} given by $u^{\mathrm{OPT}}(\theta) = \int_{\theta}^{\overline{\theta}} q^{\mathrm{OPT}}(y)dy$ for all θ.

Let q^{BM} be the *Baron–Myerson* quantity schedule, defined, for all θ, by

$$q^{\mathrm{BM}}(\theta) \equiv \arg\max_{\mathrm{q}\in[0,\bar{\mathrm{q}}]} \Big[V^\star(\mathrm{q}) - z^\star(\theta)\mathrm{q}\Big] = D^\star(z^\star(\theta)).$$

Under regularity, q^{BM} is weakly decreasing. Because q^{BM} maximizes the virtual surplus function over all weakly decreasing schedules and because robustness requires that $q(\theta) \geq \mathrm{q}_\ell$ for all θ (see Observation 1 above), the following mechanism is a natural candidate for robust optimality:

Definition 2 *The* ***Baron-Myerson-with-quantity-floor*** *mechanism* $M^\star \equiv (q^\star, u^\star)$ *is such that, for all* θ,[10]

$$q^\star(\theta) = \max\{q^{\mathrm{BM}}(\theta), \mathrm{q}_\ell\} \tag{6}$$

and $u^\star(\theta) = \int_\theta^{\overline{\theta}} q^\star(y)dy$.

Proposition 1 (optimality of Baron-Myerson-with-quantity-floor) *The mechanism* $M^\star \equiv (q^\star, u^\star)$ *is robustly optimal if and only if*

$$\int_{\underline{\theta}}^{\overline{\theta}} q^\star(y)dy \leq \int_{\underline{\theta}}^{\overline{\theta}} \underline{D}(y)dy - \int_{\underline{\theta}}^{\underline{P}(q^\star(\underline{\theta}))} \Big[\underline{D}(y) - q^\star(\underline{\theta})\Big]dy, \tag{7}$$

and, for all $\theta \in \Theta$,

$$\int_{\theta}^{\overline{\theta}} q^\star(y)dy \leq \int_{\theta}^{\overline{\theta}} \underline{D}(y)dy. \tag{8}$$

[10]Observe that $q^{\mathrm{BM}}(\underline{\theta}) = D^\star(\underline{\theta}) > \underline{D}(\underline{\theta}) > \underline{D}(\overline{\theta}) = \mathrm{q}_\ell$. On the other hand, $q^{\mathrm{BM}}(\overline{\theta}) = D^\star(z^\star(\overline{\theta}))$ can be smaller than $\underline{D}(\overline{\theta}) = \mathrm{q}_\ell$ given that $z^\star(\overline{\theta}) > \overline{\theta}$.

As we show in the Appendix, jointly, the two conditions in the proposition are equivalent to the requirement that the quantity schedule $q^\star$ satisfies all the robustness constraints in (5). Because $q^\star$ maximizes virtual surplus over all weakly decreasing schedules q satisfying $q(\theta) \geq \mathrm{q}_\ell$ for all θ, Baron-Myerson-with-quantity-floor is robustly optimal when, and only when, these conditions hold.

Note that the majorization constraints in (8) are easier to verify than the constraints in (5). In practice, it suffices to verify them at those points in which the schedule $q^\star$ crosses $\underline{D}$ from below.[11] For example, in Figure 1 below, it suffices to verify that $q^\star$, in addition to satisfying Condition (7), it satisfies Condition (8) at θ_1.

Corollary 1 (uniqueness of output schedule) *If $M^\star = (q^\star, u^\star)$ is robustly optimal, then any robustly optimal mechanism $M^{\mathrm{OPT}} = (q^{\mathrm{OPT}}, u^{\mathrm{OPT}})$ satisfies $q^{\mathrm{OPT}}(\theta) = q^\star(\theta)$ for all $\theta > \underline{\theta}$.*

Proof: When $q^\star$ solves (**ROPT**), then it also solves the relaxed program where the objective function is the same as in (**ROPT**) and the constraints in (**ROPT**) are replaced by the requirement that $q(\theta) \geq \mathrm{q}_\ell$ for all $\theta \in \Theta$ (that the constraints in (**ROPT**) imply $q(\theta) \geq \mathrm{q}_\ell$ for all $\theta \in \Theta$ follows from Observation 1). Any solution to this relaxed program must coincide with $q^\star$ at almost all θ. By continuity of $q^\star$, it must hold at all $\theta > \underline{\theta}$. ■

Corollary 2 (no demand uncertainty) *If $V^\star = \underline{V}$ (which is the case when there is no demand uncertainty, i.e., when $\mathcal{V} = \{V^\star\}$), then $M^\star = (q^\star, u^\star)$ is robustly optimal. In this case, the optimal mechanism features efficiency both at "the top" ($\underline{\theta}$) and "the bottom" ($\overline{\theta}$).*

Proof: When $V^\star = \underline{V}$ (i.e., when $D^\star = \underline{D}$), $q^{\mathrm{BM}}(\theta) \leq D^\star(\theta) = \underline{D}(\theta)$ for all θ. Because $\mathrm{q}_\ell = \underline{D}(\overline{\theta}) \leq \underline{D}(\theta)$ for all θ, we thus have that $q^\star(\theta) \leq \underline{D}(\theta)$ for all $\theta \in \Theta$, which implies

[11] In fact, Lemma 8 in Appendix D establishes that the function $\underline{W}(\cdot, q^\star)$ defined, for all $\theta \in \Theta$, by $\underline{W}(\theta, q^\star) \equiv \underline{V}(q^\star(\theta)) - \theta q^\star(\theta) - \int_{\theta}^{\overline{\theta}} q^\star(y)dy$ is weakly decreasing over an interval of types $I \subset \Theta$ if and only if $q^\star(\theta) \leq \underline{D}(\theta)$ for all $\theta \in I$. Hence, the local minima of $\underline{W}(\cdot, q^\star)$ are either $\theta \in \{\underline{\theta}, \overline{\theta}\}$, or points in which $q^\star$ crosses $\underline{D}$ from below. Once one has verified that Condition (7) holds, to verify that Condition (8) also holds, it suffices to verify that it holds at points in which the schedule $q^\star$ crosses $\underline{D}$ from below. If there are no such points, and (7) holds, then all the robustness constraints in (5) hold and $M^\star$ is robustly optimal.

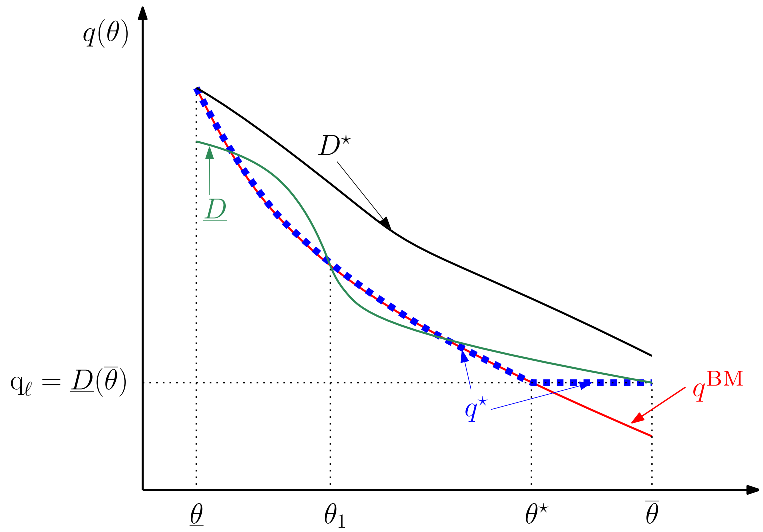


Figure 1: Baron-Myerson-with-quantity-floor.

that Condition (8) holds. Furthermore, because $q^{\mathrm{BM}}(\underline{\theta}) = D^\star(\underline{\theta}) = \underline{D}(\underline{\theta}) > \underline{D}(\overline{\theta}) = \mathrm{q}_\ell$, $q^\star(\underline{\theta}) = q^{\mathrm{BM}}(\underline{\theta}) = \underline{D}(\underline{\theta})$, which implies that Condition (7) also holds. ■

Figure 1 illustrates the output schedule $q^\star$ when $D^\star > \underline{D}$ (equivalently, when $V^\star > \underline{V}$). The upward adjustment in the quantity procured from high-cost types, relative to the Baron–Myerson benchmark, reflects the buyer's need to sustain welfare close to the efficient level when Nature assigns a higher probability to high costs than conjectured. Recall that the downward distortions in the Baron–Myerson schedule q^{BM} serve to limit the rents of low-cost types. When high-cost realizations become more likely, these distortions lose their rationale. Procuring more than q_ℓ from types in a left neighborhood of $\overline{\theta}$ would increase welfare in case Nature selects the lowest demand but would not improve the guarantee and would instead reduce welfare if the conjectured model is correct.

We now characterize the qualitative properties of robustly optimal mechanisms when they differ from the Baron–Myerson-with-quantity-floor.

Let $\theta^\star$ be the unique solution to $q^{\mathrm{BM}}(\theta^\star) = \mathrm{q}_\ell$ when such a solution exists (which is the case when, and only when, $q^{\mathrm{BM}}(\overline{\theta}) < \mathrm{q}_\ell$); else, let $\theta^\star = \overline{\theta}$. Next let $\underline{W}(\cdot, q^\star)$ be the function defined, for all $\theta \in \Theta$, by

$$\underline{W}(\theta, q^\star) \equiv \underline{V}(q^\star(\theta)) - \theta q^\star(\theta) - \int_{\theta}^{\overline{\theta}} q^\star(y)dy$$

and let

$$\theta^m \equiv \max\{\theta : \theta \in \arg\min_{y\in\Theta} \underline{W}(y, q^\star)\}.$$

Under regularity, θ^m is well defined and is the largest cost at which the function $\underline{W}(\cdot, q^\star)$ attains a minimum.[12] Type θ^m plays a key role in characterizing robustly optimal mechanisms. In particular, $\theta^m = \overline{\theta}$ if and only if the Baron–Myerson-with-quantity-floor mechanism is robustly optimal.

Proposition 2 (robust optimality: general case) *If Baron-Myerson-with-quantity-floor is not robustly optimal, then $\theta^m < \theta^\star$, and every robustly optimal mechanism $M^{\text{OPT}} = (q^{\text{OPT}}, u^{\text{OPT}})$ has the following properties:*

(a) $q^{\text{OPT}}(\theta) = q^{\text{BM}}(\theta)$ *for all* $\theta \in (\underline{\theta}, \theta^m]$;

(b) $q^{\text{OPT}}(\theta) < q^{\text{BM}}(\theta)$ *for all* $\theta \in (\theta^m, \theta^\star)$;

(c) $q^{\text{OPT}}(\theta) = \mathrm{q}_\ell$ *for all* $\theta \in [\theta^\star, \overline{\theta}]$.

Figure 2 illustrates the structure of robustly optimal quantity schedules when they differ from the quantity schedule in Baron–Myerson-with-quantity-floor. In general, robustness entails *upward adjustments* in the quantity procured from high-cost sellers and *downward adjustments* from intermediate-cost sellers, relative to the Bayesian optimum.

The upward adjustment (from $q^{\text{BM}}(\theta)$ to q_ℓ) for high-cost sellers prevents welfare losses that would arise if Nature assigns higher probability to high-cost realizations than the buyer anticipates. In contrast, the downward adjustment (from $q^{\text{BM}}(\theta)$ to $q^{\text{OPT}}(\theta) < q^{\text{BM}}(\theta)$) for intermediate-cost types limits welfare losses from over-procurement when demand turns out lower than conjectured. The benefit of these downward adjustments, however, vanish as θ approaches $\underline{\theta}$ since welfare at such low costs (under $M^\star$) exceeds G^* even when Nature selects the lowest demand $\underline{D}$.

Finally, by Lemma 10 in the Appendix, $q^\star(\theta^m) = \underline{D}(\theta^m)$. Because the quantity procured from type θ^m is efficient under the lowest demand, the only way for the buyer to satisfy the robustness constraint (4) at θ^m is to reduce the rent $u(\theta^m)$, which in turn requires lowering

[12] Under regularity, $q^\star$ is continuous. Because Θ is compact and $\underline{W}(\cdot, q^\star)$ is continuous over Θ, the set $\{\theta : \underline{W}(\theta, q^\star) \leq \underline{W}(\theta', q^\star)\ \forall\ \theta'\}$ is non-empty and compact.

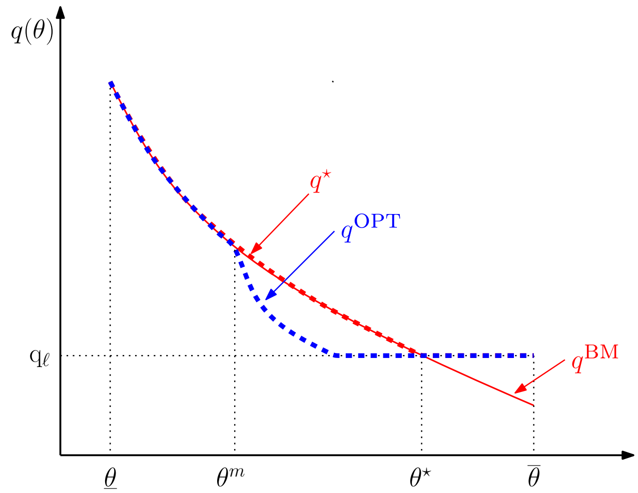


Figure 2: Illustration of Proposition 2.

the output procured from types $\theta > \theta^m$. As shown in the Appendix, once the robustness constraint is satisfied at θ^m, it is automatically satisfied for all $\theta < \theta^m$. As a result, no adjustment (vis-a-vis the Baron-Myerson quantity schedule q^{BM}) is necessary for $\theta \leq \theta^m$.

5 Monopoly regulation

Now suppose there exists a downstream market for the good supplied by the monopolist, and interpret D as the demand curve in such a market (with inverse P). In this situation, procurement mechanisms can be interpreted as a form of *quantity regulation*, in which the buyer acts as a regulator maximizing consumer surplus by controlling the quantity the monopolist supplies to consumers.[13] Hence, the analysis in Section 4 fully characterizes robust quantity regulation.

The regulator can alternatively influence market outcomes by regulating prices rather than quantities. This motivates a new class of regulatory mechanisms, which we refer to as price regulation. The latter has the advantage of allowing the quantity sold to adjust flexibly to realized demand. However, as shown in Subsection 5.2 below, it also entails disadvantages that make its overall desirability relative to quantity regulation ambiguous. Below, we first

[13]If the revenue $P(q(\theta))q(\theta)$ is accrued to the monopolist, the mechanism (q, u) is implemented through a transfer schedule $\tilde{t}(\theta) = t(\theta) - P(q(\theta))q(\theta)$, where t is the total transfer schedule corresponding to the mechanism (q, u), and where $P(q(\theta))$ is the observed market price.

characterize properties of robust price regulations and then identify conditions under which each form of regulation dominates the other.[14]

5.1 Price regulation

Price regulation consists of a price function p and a transfer function t. The price function $p : \Theta \to \mathbb{R}$ specifies the price the regulator requires the monopolist to set for each reported cost. The monopolist must then supply any quantity demanded by consumers at that price.

Because demand is uncertain, setting a price instead of a quantity exposes the monopolist to uncertainty in profits. To ensure that the monopolist participates and reports truthfully regardless of its beliefs or attitude toward uncertainty, the regulator must condition the final transfer to the monopolist on the realized demand $D \in \mathcal{D}$, which becomes known ex post. Importantly, this contingent transfer imposes no additional cost on the regulator: it neither reduces the welfare guarantee nor lowers welfare under the conjectured model $(D^\star, F^\star)$.

Definition 3 *A **price regulation** $\widetilde{M} = (p, t)$ is a pair of mappings*

$$p : \Theta \to \mathbb{R}_+ \text{ and } \; t : \Theta \times \mathcal{D} \to \mathbb{R}_+$$

where $p(\theta)$ is the price charged to consumers and $t(\theta, D)$ is the transfer to the monopolist when the cost report is θ and the realized demand is D.

We maintain that, once the regulator and the monopolist observe the quantity $D(p(\theta))$ traded in the downstream market (either concurrently with or before learning the full demand D), it is too late to make any adjustments to the price or the output supplied. As explained in the previous section, in the absence of these natural frictions, demand uncertainty is inconsequential.

The price regulation $\widetilde{M} = (p, t)$ is *ex-post incentive compatible* (EPIC) if, for all $\theta, \theta' \in \Theta$ and $D \in \mathcal{D}$,

$$t(\theta, D) - \theta D(p(\theta)) \geq t(\theta', D) - \theta D(p(\theta')).$$

[14]Under Bayesian analysis (i.e., when the regulator has a belief over $\mathcal{V} \times \mathcal{F}$ and maximizes expected consumer surplus under such a belief), price regulation always dominates quantity regulation in the sense that consumer surplus under the optimal price regulation is larger than under the optimal quantity regulation. This is not the case under robustly optimal mechanisms.

It is *ex-post individually rational* (EPIR) if, for all $\theta \in \Theta$ and $D \in \mathcal{D}$,

$$\tilde{u}(\theta, D) \equiv t(\theta, D) - \theta D(p(\theta)) \geq 0.$$

Let $\widetilde{\mathcal{M}}$ be the set of all EPIC and EPIR price regulations. For any $\widetilde{M} \in \widetilde{\mathcal{M}}$, demand $D \in \mathcal{D}$, and technology $F \in \mathcal{F}$, welfare (consumer surplus) is given by

$$\widetilde{W}(\widetilde{M}; D, F) \equiv \int \widetilde{w}(\theta, \widetilde{M}; D) F(\mathrm{d}\theta)$$

where, for all $\theta \in \Theta$, $\widetilde{M} \in \widetilde{\mathcal{M}}$, and $D \in \mathcal{D}$,

$$\widetilde{w}(\theta, \widetilde{M}; D) \equiv \int_0^{D(p(\theta))} D^{-1}(y)dy - \theta D(p(\theta)) - \tilde{u}(\theta, D).$$

The **welfare guarantee** of any price regulation $\widetilde{M} \in \widetilde{\mathcal{M}}$ is given by

$$G(\widetilde{M}) \equiv \inf_{D \in \mathcal{D}, F \in \mathcal{F}} \widetilde{W}(\widetilde{M}; D, F).$$

The **short list** of price regulations is given by

$$\widetilde{\mathcal{M}}^{\mathrm{SL}} \equiv \arg\max_{\widetilde{M} \in \widetilde{\mathcal{M}}} G(\widetilde{M}).$$

Recall that the maximal welfare guarantee for quantity regulations is $G^* \equiv \underline{V}(\mathrm{q}_\ell) - \overline{\theta}\mathrm{q}_\ell$, as shown in Lemma 1, where $\mathrm{q}_\ell \equiv \underline{D}(\overline{\theta})$ is the efficient quantity when $\theta = \overline{\theta}$ and $D = \underline{D}$ (equivalently, when $P = \underline{P}$ and $V = \underline{V}$). This same guarantee applies to price regulations:

Lemma 3 (short list of price regulations) *If $\widetilde{M} \in \widetilde{\mathcal{M}}^{\mathrm{SL}}$, then $G(\widetilde{M}) = G^*$. Moreover, $\widetilde{M} \equiv (p, t) \in \widetilde{\mathcal{M}}^{\mathrm{SL}}$ if and only if (a) p is weakly increasing, (b) for all $\theta \in \Theta$ and $D \in \mathcal{D}$,*

$$\tilde{u}(\theta, D) = \tilde{u}(\overline{\theta}, D) + \int_\theta^{\overline{\theta}} D(p(y))dy \tag{9}$$

with $\tilde{u}(\overline{\theta}, D) \geq 0$ and $\tilde{u}(\overline{\theta}, \underline{D}) = 0$, (c) $p(\overline{\theta}) = \overline{\theta}$, and (d) for all $\theta \in \Theta$ and $D \in \mathcal{D}$, $\widetilde{w}(\theta, \widetilde{M}; D) \geq G^*$.

Corollary 3 (equivalence under worst-case optimality) *The maximal welfare guarantee over all price regulations equals that under quantity regulations.*

Nature can always select the lowest possible demand $\underline{D}$ and highest cost $\overline{\theta}$. The maximal welfare at $(\underline{D}, \overline{\theta})$ is attained when the monopolist supplies the efficient output q_ℓ. Whether the regulator enforces q_ℓ by fixing the price at marginal cost or by directly imposing that quantity is immaterial. However, as shown in Subsection 5.2, identical guarantees do not imply that the two mechanisms yield the same welfare under the conjectured model.

A simple way to attain G^* is to set $p(\theta) = \overline{\theta}$ for all $\theta \in \Theta$ and $\tilde{u}(\overline{\theta}, D) = 0$ for all $D \in \mathcal{D}$. Yet many other price regulations yield the same guarantee, motivating the use of the conjectured model $(D^\star, F^\star)$ to select among them. Notably, any price regulation in the short list must set $p(\overline{\theta}) = \overline{\theta}$ and leave zero rent when $\theta = \overline{\theta}$ and $D = \underline{D}$, as only these choices achieve G^* in the worst case. Unlike with quantity regulation, however, the price function does not uniquely determine transfers, since $\tilde{u}(\overline{\theta}, D)$ may be positive for $D \neq \underline{D}$.

Definition 4 *A **Baron-Myerson-with-price-cap** regulation is a pair (p, t) such that*

$$p(\theta) = \min\{z^\star(\theta), \overline{\theta}\} \tag{10}$$

for all $\theta \in \Theta$, and the induced rent schedule $\tilde{u}(\theta, D) \equiv t(\theta, D) - \theta D(p(\theta))$ satisfies

$$\begin{aligned}
\tilde{u}(\theta, D) &= \tilde{u}(\overline{\theta}, D) + \int_\theta^{\overline{\theta}} D(p(y))dy \qquad \forall\, \theta \in \Theta,\ \forall\, D \in \mathcal{D}, \\
0 \le \tilde{u}(\overline{\theta}, D) &\le \int_0^{D(\overline{\theta})} D^{-1}(y)dy - \overline{\theta} D(\overline{\theta}) - G^* \qquad \forall\, D \in \mathcal{D} \setminus \{\underline{D}, D^\star\} \\
\tilde{u}(\overline{\theta}, \underline{D}) &= \tilde{u}(\overline{\theta}, D^\star) = 0.
\end{aligned} \tag{11}$$

All such regulations share the same price schedule, but differ in their transfer schedules.

Proposition 3 (optimality of Baron-Myerson-with-price-cap) *Every Baron–Myerson-with-price-cap regulation is robustly optimal. Moreover, in every robustly optimal price regulation $\widetilde{M}^{\mathrm{OPT}} = (\widetilde{p}^{\mathrm{OPT}}, \widetilde{u}^{\mathrm{OPT}})$, for any $\theta > \underline{\theta}$, $\widetilde{p}^{\mathrm{OPT}}(\theta) = \min\{z^\star(\theta), \overline{\theta}\}$.*

Under the conjectured model $(D^\star, F^\star)$ with gross value function $V^\star$, the virtual surplus $V^\star(D^\star(\mathrm{p})) - z^\star(\theta) D^\star(\mathrm{p})$ is quasi-concave in p and maximized at $\mathrm{p} = z^\star(\theta)$ for all $\theta \in \Theta$. The constants $\{u^{\mathrm{OPT}}(\overline{\theta}, D)\}$ can then be chosen to satisfy all the constraints in Lemma 3.

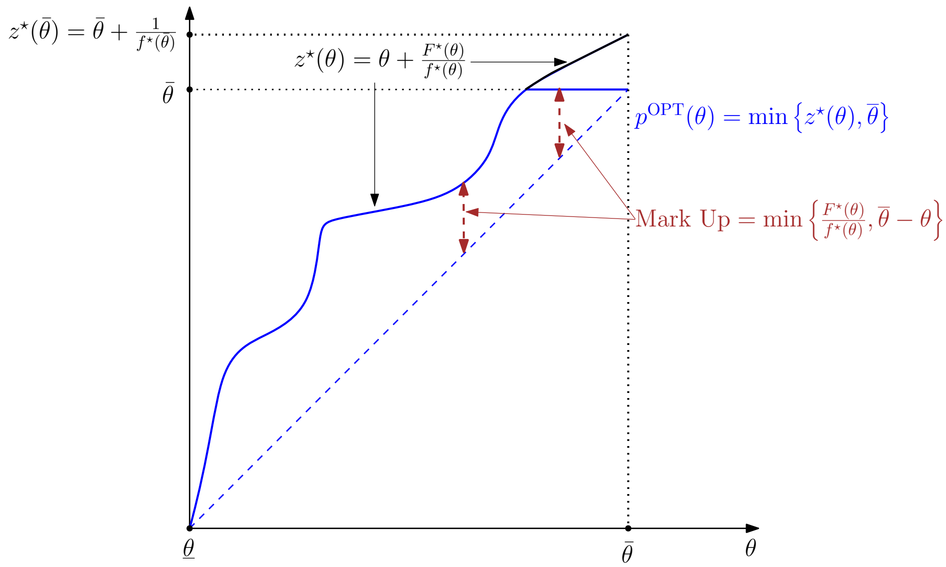


Figure 3: Robustly optimal price schedule.

Corollary 4 (robustly optimal price schedule) *The unique robustly optimal price schedule is invariant to both the conjectured demand $D^\star$ and the set of admissible demands $\mathcal{D}$. It sets a markup of $F^\star(\theta)/f^\star(\theta)$ at each cost θ, capped at $\overline{\theta}$.*

Figure 3 illustrates the result. By committing to rent payments contingent on realized demand and setting a cost-dependent markup—based solely on $F^\star$—capped at $\overline{\theta}$, the regulator maximizes welfare regardless of the uncertainty over demand or technology.

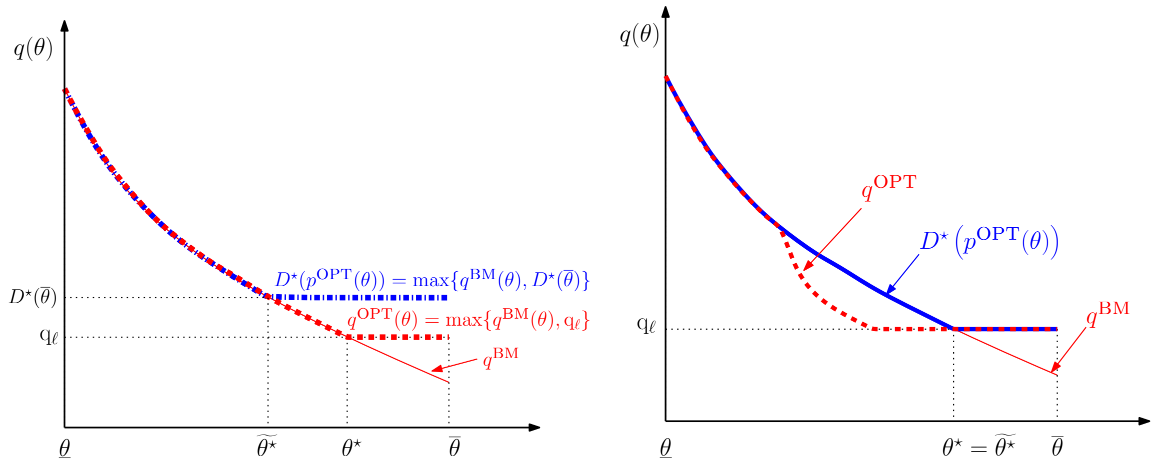
5.2 Quantity vs price regulation

Corollary 3 establishes that the maximal welfare guarantee—the highest welfare achievable under the worst-case scenario—is the same for both price and quantity regulation. However, the maximal welfare attainable under the regulator's conjectured model $(D^\star, F^\star)$ over the short list of worst-case–optimal regulations may differ between the two types of regulation.

The following definition formalizes what it means for one type of regulation to dominate the other:

Definition 5 *Price regulation dominates quantity regulation if*

$$\widetilde{W}(\widetilde{M}^{\text{OPT}}; D^\star, F^\star) \geq W(M^{\text{OPT}}; V^\star, F^\star) \tag{12}$$



A: Part 1. B: Part 2.

Figure 4: Graphical illustration of Proposition 4.

(strictly if the inequality is strict). Conversely, quantity regulation dominates price regulation if

$$W(M^{\mathrm{OPT}}; V^\star, F^\star) \geq \widetilde{W}(\widetilde{M}^{\mathrm{OPT}}; D^\star, F^\star) \tag{13}$$

(strictly if the inequality is strict). Price and quantity regulation are equivalent if (12) and (13) jointly hold.

Proposition 4 (quantity vs price regulation) *(1) If the Baron-Myerson-with-quantity-floor mechanism is robustly optimal, quantity regulation dominates price regulation (strictly if $D^\star(\overline{\theta}) > \underline{D}(\overline{\theta})$). (2) If the Baron-Myerson-with-quantity-floor mechanism is not robustly optimal and $D^\star(\overline{\theta}) = \underline{D}(\overline{\theta})$, price regulation strictly dominates quantity regulation.*

Panel A of Figure 4 illustrates the quantity schedules q^{OPT} and $D^\star(p^{\mathrm{OPT}})$ under the conjectured demand $D^\star$ for the first part of Proposition 4. Panel B illustrates the second part.

The intuition for the first part is as follows. When the Baron-Myerson-with-quantity-floor mechanism is robustly optimal, both regulations induce the monopolist to supply the same output at low costs under the conjectured model. For high costs, however, the output under the optimal price regulation can exceed that under the optimal quantity regulation. This occurs because the price is capped at $\overline{\theta}$, leading the monopolist to supply $D^\star(\overline{\theta})$ under the conjectured demand, which exceeds q_ℓ, the quantity supplied under the optimal quantity regulation, when $D^\star(\bar{\theta}) > \underline{D}(\bar{\theta})$. This oversupply increases the distance from $D^\star(z^\star(\theta))$, the Bayesian optimal quantity under the conjectured model $(D^\star, \bar{\theta})$, reducing welfare relative to quantity regulation (see Panel A of Figure 4).

For the second part, when the Baron-Myerson-with-quantity-floor is not robustly optimal, quantity regulation requires downward adjustments in output for intermediate costs to avoid over-procurement under low realized demand. These adjustments reduce welfare under the conjectured model. Price regulation, by contrast, automatically adjusts output in response to realized demand, avoiding this cost. Provided the price cap $\mathrm{p} = \bar{\theta}$ does not induce over-procurement for high costs—which is the case when $\underline{D}(\bar{\theta}) = D^\star(\bar{\theta})$—price regulation strictly dominates.

The following is an implication of the previous results and can be seen directly by combining Propositions 1 and 3.

Corollary 5 (equivalence in the absence of demand uncertainty) *If the regulator faces no demand uncertainty, price and quantity regulations are equivalent.*

Without uncertainty, it does not matter whether the regulator achieves the desired output by fixing a price or specifying a quantity. More interestingly, combining Propositions 1 and 4, one can show the following:

Corollary 6 (quantity vs price regulation: primitives) *(1) Quantity regulation strictly dominates price regulation when the downward uncertainty over demand (captured by $D^\star(\mathrm{p}) - \underline{D}(\mathrm{p})$) is concentrated at high prices and vanishing at $\underline{\theta}$ (i.e., $D^\star(\underline{\theta}) = \underline{D}(\underline{\theta})$), and, in addition, the inverse hazard rate $F^\star(\mathrm{p})/f^\star(\mathrm{p})$ is large for most $\mathrm{p} \in \Theta$, in the sense that, for all $\theta \in \Theta$,*

$$\int_{\theta}^{\bar{\theta}} \max\{D^\star(\mathrm{p} + F^\star(\mathrm{p})/f^\star(\mathrm{p})); \underline{D}(\bar{\theta})\} d\mathrm{p} \leq \int_{\theta}^{\bar{\theta}} \underline{D}(\mathrm{p}) d\mathrm{p}. \tag{14}$$

(2) Price regulation strictly dominates when downward uncertainty is concentrated at low prices and vanishing at $\bar{\theta}$ (i.e., $D^\star(\bar{\theta}) = \underline{D}(\bar{\theta})$), and, in addition, the inverse hazard rate $F^\star(\mathrm{p})/f^\star(\mathrm{p})$ is small for most $\mathrm{p} \in \Theta$, in the sense that either

$$\int_{\theta}^{\bar{\theta}} \max\{D^\star(\mathrm{p} + F^\star(\mathrm{p})/f^\star(\mathrm{p})); \underline{D}(\bar{\theta})\} d\mathrm{p} > \int_{\theta}^{\bar{\theta}} \underline{D}(\mathrm{p}) d\mathrm{p} \tag{15}$$

for some θ, or

$$\int_{\underline{\theta}}^{\overline{\theta}} \max\{D^\star(\mathrm{p} + F^\star(\mathrm{p})/f^\star(\mathrm{p})); \underline{D}(\overline{\theta})\} d\mathrm{p} > \int_{\underline{\theta}}^{\overline{\theta}} \underline{D}(\mathrm{p}) d\mathrm{p} - \int_{\underline{\theta}}^{\underline{P}(D^\star(\underline{\theta}))} \left(\underline{D}(\mathrm{p}) - D^\star(\underline{\theta})\right) d\mathrm{p}. \quad (16)$$

The corollary thus suggests that, to determine which type of regulation is preferable, the regulator should examine (a) whether uncertainty over the demand is primarily at low or high prices, and (b) whether the estimated technology (distribution $F^\star$) selects primarily low or high costs.

6 Conclusions

We study a buyer's procurement problem under model uncertainty about both the gross value of output and the seller's cost technology. The buyer first protects herself by identifying all mechanisms that maximize the welfare guarantee across a set of plausible models. Because this set generally contains multiple mechanisms, the buyer then selects the one that maximizes expected welfare under her conjectured model—estimated or calibrated from past data. This two-step approach, combining worst-case optimality with model-based evaluation, offers a disciplined and practically relevant framework for mechanism design under uncertainty.

We characterize conditions under which the robustly optimal mechanism coincides with the Bayesian mechanism, except for the presence of a quantity floor that safeguards the buyer against higher-than-expected costs. These conditions always hold when uncertainty concerns only the seller's technology. When the buyer is also uncertain about the gross value of output, robustness further requires a downward adjustment in the quantity procured from intermediate-cost sellers. Thus, the range of quantities procured by the buyer is compressed from below, relative to the Bayesian optimum.

We then extend the analysis to environments in which a continuum of atomistic consumers in a downstream market purchase from the monopolist, and study optimal regulation. Unlike standard Bayesian analyses, our results show that price regulation need not dominate quantity regulation; we then derive primitive conditions under which each regulatory instrument is optimal.

The approach and techniques appear readily portable to other design problems—for example, settings in which the designer is a seller facing uncertainty about its cost structure and/or the demand for its output. We therefore expect the findings to yield valuable insights for a broad class of design problems in which uncertainty plays a prominent role.

A Proofs of Section 3

Proof of Lemma 1. Fix any $(V, F) \in \mathcal{V} \times \mathcal{F}$, and observe that

$$W(M;V,F) = \int [V(q(\theta)) - \theta q(\theta) - u(\theta)]\, F(\mathrm{d}\theta) \geq_{(a)} \int [\underline{V}(q(\theta)) - \theta q(\theta) - u(\theta)]\, F(\mathrm{d}\theta)$$
$$\geq \inf_{\theta\in\Theta} \{\underline{V}(q(\theta)) - \theta q(\theta) - u(\theta)\},$$

where inequality (a) follows from the definition of $\underline{V}$. Hence,

$$G(M) \geq \inf_{\theta\in\Theta} \{\underline{V}(q(\theta)) - \theta q(\theta) - u(\theta)\}. \tag{17}$$

Because $\underline{V} \in \mathcal{V}$ and, for each θ, the Dirac distribution that puts probability mass one at θ is in $\mathcal{F}$, we have that, for all θ,

$$G(M) \leq \underline{V}(q(\theta)) - \theta q(\theta) - u(\theta). \tag{18}$$

Combining the inequality in (18) with the inequality in (17), we obtain Condition (2).

Finally, using (2), we obtain that

$$G(M) \leq \underline{V}(q(\overline{\theta})) - \overline{\theta} q(\overline{\theta}) - u(\overline{\theta}) \leq \underline{V}(q(\overline{\theta})) - \overline{\theta} q(\overline{\theta}) \leq \underline{V}(\mathrm{q}_\ell) - \overline{\theta}\mathrm{q}_\ell = G^*,$$

where the second inequality follows from IR and the third inequality follows from the definition of q_ℓ. This establishes (3). ■

Proof of Lemma 2. First, we show that there exists an IC and IR mechanism that delivers the welfare guarantee upper bound in (3). Consider the constant mechanism $M_L = (q_L, u_L)$ that asks each type θ to produce q_ℓ and pays $\overline{\theta}\mathrm{q}_\ell$; that is, $q_L(\theta) = \mathrm{q}_\ell$ and $t_L(\theta) = \overline{\theta}\mathrm{q}_\ell$, for every $\theta \in \Theta$ (yielding a rent $u_L(\theta) = (\overline{\theta} - \theta)\mathrm{q}_\ell$ to each θ). The mechanism M_L is clearly IC and IR. Under the mechanism M_L, when the marginal cost is θ and the gross value function is $\underline{V}$, the buyer's welfare is equal to $\underline{V}(\mathrm{q}_\ell)-\theta\mathrm{q}_\ell-u_L(\theta)=\underline{V}(\mathrm{q}_\ell)-\theta\mathrm{q}_\ell-(\overline{\theta}-\theta)\mathrm{q}_\ell=G^*$. Hence, $\inf_\theta\{\underline{V}(q(\theta))-\theta q(\theta)-u(\theta)\}=G^*$. Condition (2) in Lemma 1 then implies $G(M_L)=G^*$. By Lemma 1, we have that $M_L\in\mathcal{M}^{\mathrm{SL}}$. Condition (3) in Lemma 1 in turn implies that, for any $M\in\mathcal{M}^{\mathrm{SL}}$, $G(M)=G^*$. For a mechanism $M=(q,u)$ to be IC and IR, it must be that q is weakly decreasing and, for all θ,

$$u(\theta)=u(\overline{\theta})+\int_{\theta}^{\overline{\theta}} q(y)dy, \tag{19}$$

with $u(\overline{\theta})\geq 0$. Condition (2) in Lemma 1 in turn implies that, if $M\in\mathcal{M}^{\mathrm{SL}}$, then, for all θ

$$\underline{V}(q(\theta))-\theta q(\theta)-u(\theta)\geq G^*.$$

This is possible only if $u(\overline{\theta})=0$ (else, the constraint is violated at $\overline{\theta}$) and, for any θ, constraint (4) holds. We conclude that any $M\in\mathcal{M}^{\mathrm{SL}}$ must satisfy constraint (4).

Next, we establish that any IC and IR mechanism $M\in\mathcal{M}$ with $u(\overline{\theta})=0$ satisfying constraint (4) is in the short list. To see this, observe that, by Condition (2) in Lemma 1, $G(M)\geq G^*$. Since every mechanism in $\mathcal{M}^{\mathrm{SL}}$ has a welfare guarantee of G^*, we conclude that $M\in\mathcal{M}^{\mathrm{SL}}$. ■

B Proofs of Section 4

Proof of Proposition 1. By Observation 1, if $M=(q,u)\in\mathcal{M}^{\mathrm{SL}}$, then $q(\theta)\geq\mathrm{q}_\ell$ for all $\theta\in\Theta$. The following is thus a relaxation of the problem (**ROPT**):

$$\max_q \int_{\underline{\theta}}^{\overline{\theta}} \Big[V^\star(q(\theta))-z^\star(\theta)q(\theta)\Big]F^\star(\mathrm{d}\theta) \tag{\textbf{RP}}$$

$$\begin{aligned}\text{subject to}\quad & q\ \text{ weakly decreasing}\\ & q(\theta)\geq\mathrm{q}_\ell \qquad \forall\,\theta\in\Theta.\end{aligned}$$

That the quantity schedule $q^\star$ satisfies the constraints in the relaxed problem (**RP**) follows from the fact that, when $F^\star$ is regular, q^{BM} is decreasing and hence $q^\star$ is weakly decreasing. That $q^\star$ also satisfies the other constraint of the relaxed problem follows directly from its definition. Next, observe that, for any θ, the function $V^\star(\mathrm{q}) - z^\star(\theta)\mathrm{q}$ is concave in q and attains a maximum at $q^{\mathrm{BM}}(\theta)$. Therefore, the quantity schedule $q^\star$ maximizes the objective function in the relaxed program over all weakly decreasing functions q satisfying $q(\theta) \geq \mathrm{q}_\ell$ for all θ.

To complete the proof, it suffices to show that $q^\star$ satisfies the robustness constraints in (4) if and only if Conditions (7) and (8) in the proposition hold. We start with the following Lemma, which is proved in Appendix D.

Lemma 4 *Take any weakly decreasing function $q : \Theta \to \mathbb{R}_+$. (1) For any $\theta \in \Theta$, constraints (4) and (5) are equivalent. (2) The following statements are equivalent: (a) the inequality in (5) holds for all $\theta \in \Theta$; (b) the inequality in (5) holds for $\theta \in \{\underline{\theta}, \overline{\theta}\}$ and, for all $\theta \in (\underline{\theta}, \overline{\theta})$,*

$$\int_{\theta}^{\overline{\theta}} q(y)dy \leq \int_{\theta}^{\overline{\theta}} \underline{D}(y)dy. \tag{20}$$

We now use the lemma to complete the proof of the proposition. Suppose that $(q^\star, u^\star)$ is robustly optimal. Then $q^\star$ satisfies Condition (4) for all $\theta \in \Theta$. Part (1) of Lemma 4 implies that this is equivalent to $q^\star$ satisfying Condition (5) for all $\theta \in \Theta$. Part (2) of Lemma 4 in turn implies that $q^\star$ satisfies (7) and (8).

Conversely, suppose that $q^\star$ satisfies (7) and (8). This means that $q^\star$ satisfies (20) for all $\theta \in (\underline{\theta}, \overline{\theta})$, and, in addition satisfies Condition (5) for $\theta = \underline{\theta}$. Now observe that, if $q^\star$ satisfies Condition (20) for all $\theta \in (\underline{\theta}, \overline{\theta})$, then $q^\star(\overline{\theta}) = \mathrm{q}_\ell$. To see this, suppose that $q^\star(\overline{\theta}) > \mathrm{q}_\ell$. The continuity of $q^\star$ and $\underline{D}$ then imply that there is a left-neighborhood of $\overline{\theta}$ of positive measure where $q^\star(\theta) > \underline{D}(\theta)$. This, however, implies that Condition (20) is violated for some $\theta' < \overline{\theta}$, a contradiction. We conclude that Conditions (7) and (8) in the Proposition imply that $q^\star$ satisfies Condition (5) for $\theta \in \{\underline{\theta}, \overline{\theta}\}$ and Condition (20) for all all $\theta \in (\underline{\theta}, \overline{\theta})$. Parts (1) and (2) of Lemma 4 then jointly imply that $q^\star$ satisfies all the robustness constraints in (4). Hence, $(q^\star, u^\star)$ is robustly optimal.

■

Proof of Proposition 2. We start by establishing that, when the Baron-Myerson-with-quantity-floor mechanism is not robustly optimal, then the largest θ at which $\underline{W}(\theta, q^\star)$ reaches a minimum is strictly below θ^*.

Lemma 5 *If $M^\star = (q^\star, u^\star)$ is not robustly optimal, then $\theta^m < \overline{\theta}$. Moreover, $\theta^m < \theta^\star$ and $q^\star(\theta^m) = q^{\mathrm{BM}}(\theta^m)$.*

Proof: We first establish that $\theta^m < \overline{\theta}$. Assume, towards a contradiction, that $\theta^m = \overline{\theta}$. Then, $\underline{W}(\theta, q^\star) \geq \underline{W}(\overline{\theta}, q^\star)$ for every $\theta \in \Theta$. Moreover, we must have that $q^\star(\overline{\theta}) = \mathrm{q}_\ell$. To see this, suppose $q^\star(\overline{\theta}) > \mathrm{q}_\ell$. The continuity of $q^\star$ and $\underline{D}$ then implies that there exists a left-neighborhood $(\theta_1, \overline{\theta}]$ of $\overline{\theta}$ of positive Lebesgue measure in which $q^\star(\theta) > \underline{D}(\theta)$. Lemma 8 (in Appendix D) then implies that $\underline{W}(\cdot, q^\star)$ is increasing over $(\theta_1, \overline{\theta}]$, a contradiction to $\theta^m = \overline{\theta}$. Thus $q^\star(\overline{\theta}) = \mathrm{q}_\ell$ and $\underline{W}(\overline{\theta}, q^\star) = G^*$. That $\theta^m = \overline{\theta}$ then implies that $\underline{W}(\theta, q^\star) \geq G^*$ for every θ. The schedule $q^\star$ thus satisfies all the robustness constraints in (4). As established in the proof of Proposition 1, $q^\star$ also solves the relaxed problem (**RP**). Therefore, $M^\star = (q^\star, u^\star)$ must be robustly optimal, a contradiction.

Next, we show that $\theta^m < \theta^\star$ and $q^\star(\theta^m) = q^{\mathrm{BM}}(\theta^m)$. If $\theta^\star < \overline{\theta}$, then for every $\theta \geq \theta^\star$, $q^\star(\theta) = \mathrm{q}_\ell$ and $\underline{W}(\theta, q^\star) = G^*$. Because $\underline{W}(\theta^m, q^\star) < G^*$, it must be that $\theta^m < \theta^\star$. Next suppose that $\theta^\star = \overline{\theta}$. Because $\theta^m < \overline{\theta}$, we thus have that $\theta^m < \theta^\star$. That $\theta^m < \theta^\star$ in turn implies that $q^\star(\theta^m) = q^{\mathrm{BM}}(\theta^m)$. ■

Now let

$$\mathcal{Q} := \{q : [\underline{\theta}, \overline{\theta}] \to [\mathrm{q}_\ell, \bar{\mathrm{q}}] : q \text{ weakly decreasing and } q(\overline{\theta}) = \mathrm{q}_\ell\},$$

and note that $\mathcal{Q}$ is a convex set. Using Lemma 4, we have that the optimal quantity schedule q^{OPT} solves the following problem.

$$\max_{q \in \mathcal{Q}} \int \Big[V^\star(q(\theta)) - z^\star(\theta) q(\theta)\Big] f^\star(\theta) \mathrm{d}\theta$$

$$\text{subject to} \tag{FP}$$

$$\int_\theta^{\overline{\theta}} \underline{D}(y) \mathrm{d}y \geq \int_\theta^{\overline{\theta}} q(y) \mathrm{d}y \qquad \forall\, \theta \in (\underline{\theta}, \overline{\theta}) \tag{Maj}$$

$$\underline{V}(q(\underline{\theta})) - \underline{\theta} q(\underline{\theta}) - \int_{\underline{\theta}}^{\overline{\theta}} q(y)\mathrm{d}y \geq G^*. \qquad \textbf{(Rob-low)}$$

Note that since every $q \in \mathcal{Q}$ satisfies $q(\overline{\theta}) = \mathrm{q}_\ell$, the robustness constraint is satisfied at $\theta = \overline{\theta}$, which explains why it is not explicitly included in (**FP**).

For any $\theta \in (\underline{\theta}, \overline{\theta})$, let $\lambda(\theta) \geq 0$ be the Lagrange multiplier associated with the constraint in (**Maj**). Similarly, let $\mu \geq 0$ be the Lagrange multiplier associated with the constraint in (**Rob-low**). For any $q \in \mathcal{Q}$, $\lambda : (\underline{\theta}, \overline{\theta}) \to \mathbb{R}_+$, and $\mu \geq 0$, let

$$\begin{aligned} \mathcal{L}(q; \lambda, \mu) :=& \int_{\underline{\theta}}^{\overline{\theta}} \Big[V^\star(q(\theta)) - z^\star(\theta) q(\theta) \Big] f^\star(\theta) \mathrm{d}\theta \\ &+ \int_{\underline{\theta}}^{\overline{\theta}} \lambda(\theta) \left[\int_{\theta}^{\overline{\theta}} \left(\underline{D}(y) - q(y) \right) \mathrm{d}y \right] \mathrm{d}\theta \\ &+ \mu \Big[\underline{V}(q(\underline{\theta})) - \underline{\theta} q(\underline{\theta}) - \int_{\underline{\theta}}^{\overline{\theta}} q(y)\mathrm{d}y - G^* \Big] \end{aligned}$$

be the Lagrangean function associated with the above optimization problem.

For any $\theta \in [\underline{\theta}, \overline{\theta}]$, let $\Lambda(\theta; \lambda) := \int_{\underline{\theta}}^{\theta} \lambda(y)\, dy$,[15] and note that

$$\int_{\underline{\theta}}^{\overline{\theta}} \lambda(\theta) \left[\int_{\theta}^{\overline{\theta}} \left(\underline{D}(y) - q(y) \right) \mathrm{d}y \right] \mathrm{d}\theta = \int_{\underline{\theta}}^{\overline{\theta}} \Lambda(\theta; \lambda) \big[\underline{D}(\theta) - q(\theta) \big] \mathrm{d}\theta.$$

Using Λ, the Lagrangian function $\mathcal{L}(q; \lambda, \mu)$ can be rewritten as

$$\begin{aligned} \mathcal{L}(q; \lambda, \mu) :=& \int_{\underline{\theta}}^{\overline{\theta}} \Big[V^\star(q(\theta)) - z^\star(\theta) q(\theta) \Big] f^\star(\theta) \mathrm{d}\theta \\ &+ \int_{\underline{\theta}}^{\overline{\theta}} \Lambda(\theta; \lambda) \Big[\underline{D}(\theta) - q(\theta) \Big] \mathrm{d}\theta + \mu \Big[\underline{V}(q(\underline{\theta})) - \underline{\theta} q(\underline{\theta}) - \int_{\underline{\theta}}^{\overline{\theta}} q(y)\mathrm{d}y - G^* \Big] \end{aligned}$$

[15]Even if λ is defined only for $\theta \in (\underline{\theta}, \overline{\theta})$, $\Lambda(\theta; \lambda)$ is defined also for $\theta \in \{\underline{\theta}, \overline{\theta}\}$.

$$= \int_{\underline{\theta}}^{\overline{\theta}} \Big[V^\star(q(\theta)) - \Big(z^\star(\theta) + \frac{\Lambda(\theta;\lambda)+\mu}{f^\star(\theta)}\Big)q(\theta)\Big] f^\star(\theta)\mathrm{d}\theta$$

$$+ \int_{\underline{\theta}}^{\overline{\theta}} \Lambda(\theta;\lambda)\underline{D}(\theta)\mathrm{d}\theta + \mu\Big[\underline{V}(q(\underline{\theta})) - \underline{\theta}q(\underline{\theta}) - G^*\Big]. \tag{21}$$

Note that the second term in (21) is independent of q and the last term depends on q only through $q(\underline{\theta})$. The proof below takes advantage of these two properties.

We now apply the necessity theorem of Luenberger (1997) (Theorem 1, Page 217). To apply this theorem, we let $\tilde{\mathcal{Q}} := \{q : [\underline{\theta}, \overline{\theta}) \to [\mathrm{q}_\ell, \bar{\mathrm{q}}] : q \text{ is weakly decreasing}\}$. Note that the only distinction with respect to $\mathcal{Q}$ is that the policies in $\tilde{\mathcal{Q}}$ are not defined for $\theta = \overline{\theta}$. However, because any policy in $\tilde{\mathcal{Q}}$ satisfies $q(\theta) \geq \mathrm{q}_\ell$ all $\theta \in [\underline{\theta}, \overline{\theta})$ and because robustness requires that $q(\overline{\theta}) = \mathrm{q}_\ell$, such a redefinition is inconsequential and only serves the purpose of guaranteeing that the constraint qualification in Luenberger's theorem is satisfied. The theorem then implies that there exists a function $\lambda^{\text{OPT}} : (\underline{\theta}, \overline{\theta}) \to \mathbb{R}_+$ and a scalar $\mu^{\text{OPT}} \geq 0$ such that

$$q^{\text{OPT}} \in \arg\max_{q \in \tilde{\mathcal{Q}}} \mathcal{L}(q; \lambda^{\text{OPT}}, \mu^{\text{OPT}}), \tag{22}$$

$$\int_{\underline{\theta}}^{\overline{\theta}} \lambda^{\text{OPT}}(\theta) \left[\int_{\theta}^{\overline{\theta}} \big(\underline{D}(y) - q^{\text{OPT}}(y)\big)\,\mathrm{d}y\right] \mathrm{d}\theta = 0, \tag{23}$$

$$\mu^{\text{OPT}}\Big[\underline{V}(q^{\text{OPT}}(\underline{\theta})) - \underline{\theta}q^{\text{OPT}}(\underline{\theta}) - G^*\Big] = 0. \tag{24}$$

Now, given $q^{\text{OPT}}(\underline{\theta})$, let

$$\mathcal{Q}^-(q^{\text{OPT}}(\underline{\theta})) := \{q : (\underline{\theta}, \overline{\theta}) \to [\mathrm{q}_\ell, q^{\text{OPT}}(\underline{\theta})] : \; q \text{ weakly decreasing}\}.$$

That q^{OPT} satisfies (22), along with the fact that the last two terms of (21) do not depend on the value that q takes over $(\underline{\theta}, \overline{\theta})$ implies that q^{OPT} must attain the supremum of the following auxiliary problem, indexed by $q^{\text{OPT}}(\underline{\theta})$:

$$\sup_{q \in \mathcal{Q}^-(q^{\text{OPT}}(\underline{\theta}))} \int_{\underline{\theta}}^{\overline{\theta}} \Big[V^\star(q(\theta)) - \Big(z^\star(\theta) + \frac{\Lambda^{\text{OPT}}(\theta) + \mu^{\text{OPT}}}{f^\star(\theta)}\Big)q(\theta)\Big] f^\star(\theta)\mathrm{d}\theta, \qquad \textbf{(UNCONST)}$$

where, for any $\theta \in (\underline{\theta}, \overline{\theta})$,

$$\Lambda^{\mathrm{OPT}}(\theta) := \Lambda(\theta; \lambda^{\mathrm{OPT}}).$$

The term in square brackets in the integrand in (**UNCONST**) is separable in $q(\theta)$ and θ, in the sense of Toikka (2011). The solution can thus be obtained by ironing the modified virtual cost

$$z^M(\theta) := z^\star(\theta) + \frac{\Lambda^{\mathrm{OPT}}(\theta) + \mu^{\mathrm{OPT}}}{f^\star(\theta)} \qquad \forall\, \theta \in [\underline{\theta}, \overline{\theta}]$$

as follows. For any $\phi \in [0,1]$, let

$$h(\phi) := z^M\big((F^\star)^{-1}(\phi)\big), \qquad H(\phi) := \int_0^\phi h(\tilde{\phi})\, \mathrm{d}\tilde{\phi}.$$

Let $\overline{H} := \mathrm{conv} H$ be the convex hull of H, i.e., the highest convex function on $[0,1]$ such that $\overline{H} \leqslant H$. Because $\overline{H}$ is convex, it is continuously differentiable, except at possibly countably many points. At all ϕ at which $\overline{H}$ is differentiable, let

$$\overline{h}(\phi) := \overline{H}'(\phi)$$

and then extend $\overline{h}$ to all $[0,1]$ by right continuity; that is, at any ϕ at which $\overline{H}'(\phi)$ does not exist, let $\overline{h}(\phi)$ be the right-derivative of $\overline{H}$ at ϕ. For any $\theta \in [\underline{\theta}, \overline{\theta}]$, then let

$$\bar{z}^M(\theta) \equiv \overline{h}(F^\star(\theta))$$

be the *ironed modified virtual cost*. Theorem 3.7 in Toikka (2011) then implies that q^{OPT} attains the supremum in (**UNCONST**) if and only if

(a) for almost all $\theta \in (\underline{\theta}, \overline{\theta})$,

$$q^{\mathrm{OPT}}(\theta) \in \arg \max_{\mathrm{q} \in [\mathrm{q}_\ell, q^{\mathrm{OPT}}(\underline{\theta})]} \Big[V^\star(\mathrm{q}) - \bar{z}^M(\theta)\mathrm{q} \Big],$$

(b) for all open intervals $I \subset \Theta$ such that $\overline{H}(F^\star(\theta)) < H(F^\star(\theta))$ for all $\theta \in I$, q^{OPT} is constant over I (pooling property).

For any $\theta \in \Theta$, the function $V^\star(\mathrm{q}) - \bar{z}^M(\theta)\mathrm{q}$ is strictly concave in q, with a unique maximum at

$$\breve{q}(\theta) := D^\star(\bar{z}^M(\theta)).$$

Because $\bar{z}^M$ is weakly increasing, $\breve{q}$ is weakly decreasing. Hence, for all $\theta \in (\underline{\theta}, \overline{\theta})$,

$$q^{\mathrm{OPT}}(\theta) = \begin{cases} \mathrm{q}_\ell & \text{if } \breve{q}(\theta) < \mathrm{q}_\ell, \\ \breve{q}(\theta) & \text{if } \breve{q}(\theta) \in [\mathrm{q}_\ell, q^{\mathrm{OPT}}(\underline{\theta})], \\ q^{\mathrm{OPT}}(\underline{\theta}) & \text{if } \breve{q}(\theta) > q^{\mathrm{OPT}}(\underline{\theta}). \end{cases} \tag{25}$$

Now let $\theta_{\inf} := \inf\{\theta \in \Theta : \Lambda^{\mathrm{OPT}}(\theta) + \mu^{\mathrm{OPT}} > 0\}$. Note that $\{\theta \in \Theta : \Lambda^{\mathrm{OPT}}(\theta) + \mu^{\mathrm{OPT}} > 0\} \neq \emptyset$. To see this, observe that, if $\Lambda^{\mathrm{OPT}}(\theta) + \mu^{\mathrm{OPT}} = 0$ for all $\theta \in \Theta$, then the BM-with-quantity-floor schedule $q^\star$ satisfies the optimality condition (22) and hence it is robustly optimal. This contradicts the assumption that BM-with-quantity-floor is not robustly optimal. Moreover,

$$\theta_{\inf} < \theta^\star. \tag{26}$$

To see this, note that, when $\theta_{\inf} \geq \theta^\star$ then $\mu^{\mathrm{OPT}} = 0$. But then again the BM-with-quantity-floor schedule $q^\star$ satisfies the optimality condition (22) and hence it is robustly optimal, contradicting the assumption that it is not.

The rest of the proof is in two steps. Step 1 establishes that

$$\begin{aligned} &q^{\mathrm{OPT}}(\theta) = q^{\mathrm{BM}}(\theta) \qquad \forall\, \theta \in (\underline{\theta}, \theta_{\inf}], \\ &q^{\mathrm{OPT}}(\theta) < q^{\mathrm{BM}}(\theta) \qquad \forall\, \theta \in (\theta_{\inf}, \theta^\star), \\ &q^{\mathrm{OPT}}(\theta) = \mathrm{q}_\ell \qquad \forall\, \theta \in [\theta^\star, \overline{\theta}]. \end{aligned}$$

Step 2 establishes that $\theta_{\inf} = \theta^m$.

Step 1. Observe that the definitions of z^M and $\theta_{\inf}$, along with the monotonicity of Λ^{OPT} implies that $z^M(\theta) = z^\star(\theta)$ for all $\theta \in (\underline{\theta}, \theta_{\inf})$, whereas $z^M(\theta) > z^\star(\theta)$ for all $\theta > \theta_{\inf}$. To prove the result in Step 1, we use Lemma 9 in Appendix D, which establishes that a similar relation holds between $\bar{z}^M$ and $z^\star$. Namely, $\bar{z}^M(\theta) = z^\star(\theta)$ for all $\theta \in (\underline{\theta}, \theta_{\inf})$, and $\bar{z}^M(\theta) > z^\star(\theta)$ for all $\theta \in (\theta_{\inf}, \theta^\star)$.

To see how this lemma implies the result in Step 1, start by picking any $\theta \in (\theta_{\inf}, \theta^\star)$. By Lemma 9, $\bar{z}^M(\theta) > z^\star(\theta)$, which implies that $\breve{q}(\theta) < q^{\mathrm{BM}}(\theta)$. Condition (25) then implies that, no matter whether $q^{\mathrm{OPT}}(\theta) = \mathrm{q}_\ell$, $q^{\mathrm{OPT}}(\theta) = \breve{q}(\theta)$, or $q^{\mathrm{OPT}}(\theta) = q^{\mathrm{OPT}}(\underline{\theta})$, $q^{\mathrm{OPT}}(\theta) < q^{\mathrm{BM}}(\theta)$.

Next consider the interval $[\theta^\star, \overline{\theta}]$. By Lemma 9, there exists $\epsilon > 0$ such that $\bar{z}^M(\theta) > z^\star(\theta)$ for all $\theta \in (\theta^\star - \epsilon, \theta^\star)$. Because $z^\star$ is increasing and continuous, and $\bar{z}^M$ is weakly increasing and right-continuous, we have that $\bar{z}^M(\theta^\star) \geq z^\star(\theta^\star)$. This further implies that $\bar{z}^M(\theta) \geq z^\star(\theta^\star)$ for all $\theta \in [\theta^\star, \overline{\theta}]$ because $\bar{z}^M$ is weakly increasing. As a result, for all $\theta \in [\theta^\star, \overline{\theta}]$, $\breve{q}(\theta) = D^\star(\bar{z}(\theta)) \leq D^\star(z^\star(\theta^\star)) = \mathrm{q}_\ell$. Condition (25) then implies that $q^{\mathrm{OPT}}(\theta) = \mathrm{q}_\ell$ for all $\theta \in [\theta^\star, \overline{\theta}]$.

Finally, consider the interval $(\underline{\theta}, \theta_{\inf}]$. When $\theta_{\inf} = \underline{\theta}$ the result holds vacuously. Thus suppose that $\theta_{\inf} > \underline{\theta}$. The definition of $\theta_{\inf}$ then implies that $\Lambda^{\mathrm{OPT}}(\theta) + \mu^{\mathrm{OPT}} = 0$ for all $\theta \in (\underline{\theta}, \theta_{\inf})$. That q^{OPT} satisfies (22) then implies that $q^{\mathrm{OPT}}(\theta) = q^{\mathrm{BM}}(\theta)$ for all $\theta \in (\underline{\theta}, \theta_{\inf})$. To see this, suppose that this is note true and consider the quantity schedule

$$q(\theta) = \begin{cases} q^{\mathrm{BM}}(\theta) & \text{if } \theta \in [\underline{\theta}, \theta_{\inf}] \\ q^{\mathrm{OPT}}(\theta) & \text{if } \theta \in (\theta_{\inf}, \overline{\theta}]. \end{cases}$$

Note that the schedule q is weakly increasing because (a) both q^{BM} and q^{OPT} are weakly decreasing, (b) q^{BM} is continuous, and (c) $q^{\mathrm{OPT}}(\theta) < q^{\mathrm{BM}}(\theta)$ for all $\theta \in (\theta_{\inf}, \theta^\star)$, as established above. But then $\mathcal{L}(q; \lambda^{\mathrm{OPT}}, \mu^{\mathrm{OPT}}) > \mathcal{L}(q^{\mathrm{OPT}}; \lambda^{\mathrm{OPT}}, \mu^{\mathrm{OPT}})$, contradicting the fact that q^{OPT} satisfies Condition (22). This completes the proof of the Step 1.

Step 2. We now prove that $\theta_{\inf} = \theta^m$. From Lemma 5, we know that $\theta^m < \theta^\star$. The next lemma leverages this property to establishes that $\theta^m \leq \theta_{\inf}$.

Lemma 6 *If $\theta^m < \theta^\star$, then $\theta_{\inf} \geq \theta^m$.*

Proof: Assume, towards a contradiction, that $\theta_{\inf} < \theta^m$. Thus, $\underline{\theta} \leq \theta_{\inf} < \theta^m < \theta^\star$. The result in Step 1 in the proof of this proposition then implies that $q^{\mathrm{OPT}}(\theta) < q^{\mathrm{BM}}(\theta)$ for all $\theta \in (\theta_{\inf}, \theta^m]$. Then, let $\widetilde{M} = (\tilde{q}, \tilde{u})$ be the mechanism where the quantity schedule is given by

$$\tilde{q}(\theta) = \begin{cases} q^{\mathrm{OPT}}(\theta) & \text{if } \ \theta \in [\underline{\theta}, \theta_{\inf}) \\ q^{\mathrm{BM}}(\theta) & \text{if } \ \theta \in [\theta_{\inf}, \theta^m] \\ q^{\mathrm{OPT}}(\theta) & \text{otherwise} \end{cases}$$

and where $\tilde{u}(\theta) = \int_\theta^{\bar{\theta}} \tilde{q}(y)dy$ for all $\theta \in \Theta$. The result in Step 1 in the proof of this proposition implies that $q^{\mathrm{OPT}}(\theta) = q^{\mathrm{BM}}(\theta)$ for $\theta \in [\underline{\theta}, \theta_{\inf})$, and therefore, $\tilde{q}$ is weakly decreasing. Because $\tilde{q}$ is weakly decreasing and $\tilde{u}$ satisfies the envelope formula, $\widetilde{M}$ is IC and IR. Below, we first show that $\widetilde{M}$ yields a higher welfare to the buyer than M^{OPT}, and then that $\widetilde{M} \in \mathcal{M}^{\mathrm{SL}}$, contradicting the optimality of M^{OPT}.

Under the model $(V^\star, F^\star)$, the buyer's expected welfare when the selected mechanism is $\widetilde{M}$ is

$$\int_{\underline{\theta}}^{\overline{\theta}} \Big[V^\star(\tilde{q}(\theta)) - z^\star(\theta)\tilde{q}(\theta)\Big] F^\star(\mathrm{d}\theta)$$

whereas it is equal to

$$\int_{\underline{\theta}}^{\overline{\theta}} \Big[V^\star(q^{\mathrm{OPT}}(\theta)) - z^\star(\theta)q^{\mathrm{OPT}}(\theta)\Big] F^\star(\mathrm{d}\theta)$$

when the selected mechanism is M^{OPT}. Than the first integral is greater follows from the fact that, for any θ, $V^\star(\mathrm{q}) - z^\star(\theta)\mathrm{q}$ is strictly concave in q reaching a maximum at $\mathrm{q} = q^{\mathrm{BM}}(\theta)$.

We now show that $\tilde{q}$ satisfies the robustness constraints. Clearly, this is true for any $\theta > \theta^m$, because, for any such θ, $\underline{W}(\theta, \tilde{q}) = \underline{W}(\theta, q^{\mathrm{OPT}})$. Thus consider $\theta \in [\underline{\theta}, \theta^m]$. Because for any $\theta \leq \theta^m$, $\tilde{q}(\theta) = q^{\mathrm{BM}}(\theta) = q^\star(\theta)$, we have that, for any $\theta \leq \theta^m$,

$$\begin{aligned}
\underline{W}(\theta, \tilde{q}) - \underline{W}(\theta, q^\star) = & \int_{\theta^m}^{\overline{\theta}} q^\star(y)dy - \int_{\theta^m}^{\overline{\theta}} q^{\mathrm{OPT}}(y)dy \\
& \geq_{(a)} \Big[\underline{V}(q^{\mathrm{OPT}}(\theta^m)) - \theta^m q^{\mathrm{OPT}}(\theta^m)\Big] - \Big[\underline{V}(q^\star(\theta^m)) - \theta^m q^\star(\theta^m)\Big] \\
& \quad + \int_{\theta^m}^{\overline{\theta}} q^\star(y)dy - \int_{\theta^m}^{\overline{\theta}} q^{\mathrm{OPT}}(y)dy \\
& = \underline{W}(\theta^m, q^{\mathrm{OPT}}) - \underline{W}(\theta^m, q^\star).
\end{aligned}$$

Inequality (a) follows from the fact that $\underline{D}(\theta^m)$ maximizes $\underline{V}(\mathrm{q}) - \theta^m\mathrm{q}$ over $\mathbb{R}_+$ along with the facts that (a) $q^{\mathrm{BM}}(\theta^m) = q^\star(\theta^m)$ (by the fact that $\theta^m < \theta^\star$) and (b) $q^\star(\theta^m) = \underline{D}(\theta^m)$ (by Lemma 10, stated and proved in Appendix D, along with the fact that $\theta^m > \underline{\theta}$). Therefore, for any $\theta \leq \theta^m$, $\underline{W}(\theta, \tilde{q}) - \underline{W}(\theta^m, q^{\mathrm{OPT}}) \geq \underline{W}(\theta, q^\star) - \underline{W}(\theta^m, q^\star) \geq 0$, where the last inequality follows from the definition of θ^m. Because q^{OPT} satisfies the robustness constraints, $\underline{W}(\theta^m, q^{\mathrm{OPT}}) \geq G^*$. Hence, $\underline{W}(\theta, \tilde{q}) \geq G^*$. We conclude that $\widetilde{M} \in \mathcal{M}^{\mathrm{SL}}$ and yields a strictly higher expected welfare to the buyer than M^{OPT}, contradicting the optimality of M^{OPT}. ■

The next lemma shows that, if $\theta_{\inf} > \underline{\theta}$, the robustness constraint binds at $\theta_{\inf}$.

Lemma 7 *Suppose $\theta_{\inf} > \underline{\theta}$. Then $\underline{W}(\theta_{\inf}, q^{\mathrm{OPT}}) = G^*$.*

Proof: By definition of $\theta_{\inf}$, for all $\theta < \theta_{\inf}$, $\mu^{\mathrm{OPT}} + \Lambda^{\mathrm{OPT}}(\theta) = 0$, which implies that $\mu^{\mathrm{OPT}} = 0$. Furthermore, there exists $\delta > 0$ such that $\lambda^{\mathrm{OPT}}(\theta) > 0$ for almost all $\theta \in (\theta_{\inf}, \theta_{\inf} + \delta)$. Now, define

$$S(\theta, q^{\mathrm{OPT}}) := \int_{\theta}^{\bar{\theta}} \left(\underline{D}(y) - q^{\mathrm{OPT}}(y)\right) \mathrm{d}y \qquad \forall\, \theta \in \Theta.$$

Because q^{OPT} satisfies (**Maj**), $S(\theta, q^{\mathrm{OPT}}) \geq 0$ for all $\theta \in (\underline{\theta}, \bar{\theta})$. We first argue that $S(\theta_{\inf}, q^{\mathrm{OPT}}) = 0$. To do so, observe that the complementary slackness conditions associated with the (**Maj**) constraint (23) imply that

$$\int_{\underline{\theta}}^{\bar{\theta}} \lambda^{\mathrm{OPT}}(\theta) S(\theta, q^{\mathrm{OPT}}) \mathrm{d}\theta = 0.$$

Because, for all $\theta \in (\underline{\theta}, \bar{\theta})$, $S(\theta, q^{\mathrm{OPT}}) \geq 0$ and $\lambda^{\mathrm{OPT}}(\theta) \geq 0$, the complementary slackness condition is satisfied by q^{OPT} if and only if $\lambda^{\mathrm{OPT}}(\theta) S(\theta, q^{\mathrm{OPT}}) = 0$ for almost all $\theta \in \Theta$. Because $\lambda^{\mathrm{OPT}}(\theta) > 0$ for almost all $\theta \in (\theta_{\inf}, \theta_{\inf} + \delta)$, we thus have that $S(\theta, q^{\mathrm{OPT}}) = 0$ for almost all $\theta \in (\theta_{\inf}, \theta_{\inf} + \delta)$. This means that there exists a sequence $\{\theta_n\}$, with $\theta_n > \theta_{\inf}$ such that $\lim_{n\to\infty} \theta_n = \theta_{\inf}$ and $S(\theta_n, q^{\mathrm{OPT}}) = 0$ for all n. By continuity of S, we thus have that $S(\theta_{\inf}, q^{\mathrm{OPT}}) = 0$.

Next observe that Lemma 4 implies that

$$\underline{W}(\theta_{\inf}, q^{\mathrm{OPT}}) - G^* = S(\theta_{\inf}, q^{\mathrm{OPT}}) - \int_{\theta_{\inf}}^{\underline{P}(q^{\mathrm{OPT}}(\theta_{\inf}))} \left(\underline{D}(y) - q^{\mathrm{OPT}}(\theta_{\inf})\right) \mathrm{d}y. \tag{27}$$

The last term in the above expression is the dead-weight loss of procuring $q^{\mathrm{OPT}}(\theta_{\inf})$ instead of $\underline{D}(\theta_{\inf})$ when the marginal cost is $\theta_{\inf}$ and the demand is $\underline{D}$. Because this term is non-negative and because $S(\theta_{\inf}, q^{\mathrm{OPT}}) = 0$, we conclude that $\underline{W}(\theta_{\inf}, q^{\mathrm{OPT}}) \leq G^*$. Because q^{OPT} satisfies the robustness constraints, $\underline{W}(\theta_{\inf}, q^{\mathrm{OPT}}) \geq G^*$. We thus conclude that $\underline{W}(\theta_{\inf}, q^{\mathrm{OPT}}) = G^*$.
■

We now show that the last two lemmas imply that $\theta^m = \theta_{\inf}$. From Lemma 6, we already know that $\theta_{\inf} \geq \theta_m$. Now suppose that $\theta_{\inf} > \theta_m$. Then, inequality (26) implies that $\underline{\theta} \leq \theta^m < \theta_{\inf} < \theta^\star$. Now, observe that

$$\underline{W}(\theta_{\inf}, q^{\mathrm{OPT}}) - \underline{W}(\theta^m, q^{\mathrm{OPT}}) = \left[\underline{V}(q^{\mathrm{OPT}}(\theta_{\inf})) - \theta_{\inf} q^{\mathrm{OPT}}(\theta_{\inf})\right] - \left[\underline{V}(q^{\mathrm{OPT}}(\theta^m)) - \theta^m q^{\mathrm{OPT}}(\theta^m)\right]$$

$$
\begin{aligned}
&+\int_{\theta^m}^{\theta_{\text{inf}}} q^{\text{OPT}}(y)\mathrm{d}y \\
&= \left[\underline{V}(q^{\text{BM}}(\theta_{\text{inf}})) - \theta_{\text{inf}} q^{\text{BM}}(\theta_{\text{inf}})\right] - \left[\underline{V}(q^{\text{BM}}(\theta^m)) - \theta^m q^{\text{BM}}(\theta^m)\right] \\
&+\int_{\theta^m}^{\theta_{\text{inf}}} q^{\text{BM}}(y)\mathrm{d}y \\
&= \underline{W}(\theta_{\text{inf}}, q^\star) - \underline{W}(\theta^m, q^\star) \\
&> 0.
\end{aligned}
$$

The second equality follows from the fact that $q^{\text{OPT}}(\theta) = q^{\text{BM}}(\theta)$ for all $\theta \in (\underline{\theta}, \theta_{\text{inf}}]$, as establish in Step 1. The inequality follows from the definition of θ^m and the fact that $\theta^m < \theta_{\text{inf}}$. But this implies that $\underline{W}(\theta_{\text{inf}}, q^{\text{OPT}}) > \underline{W}(\theta^m, q^{\text{OPT}}) \geq G^*$, where the inequality follows from the fact that q^{OPT} satisfies the robustness constraints. However, that $\underline{W}(\theta_{\text{inf}}, q^{\text{OPT}}) > G^*$ contradicts Lemma 7, which establishes that if $\theta_{\text{inf}} > \underline{\theta}$, then $\underline{W}(\theta_{\text{inf}}, q^{\text{OPT}}) = G^*$.

We thus conclude that $\theta_{\text{inf}} = \theta_m$, as claimed. This concludes the proof of the proposition.

■

C Proofs of Section 5

Proof of Lemma 3. First, we show that $G(\widetilde{M}) \leq G^*$ for any $\widetilde{M} \equiv (p, t) \in \widetilde{\mathcal{M}}$. To establish this, we equivalently represent the mechanism as $\widetilde{M} \equiv (p, \tilde{u})$, where $\tilde{u}(\theta, D) \equiv t(\theta, D) - \theta D(p(\theta))$ for all $\theta \in \Theta$ and $D \in \mathcal{D}$. Observe that, for any $\widetilde{M} \in \widetilde{\mathcal{M}}$,

$$
\begin{aligned}
G(\widetilde{M}) &\leq_{(i)} \underline{V}(\underline{D}(p(\overline{\theta}))) - \overline{\theta}\underline{D}(p(\overline{\theta})) - \tilde{u}(\overline{\theta}, \underline{D}) \\
&\leq_{(ii)} \underline{V}(\underline{D}(p(\overline{\theta}))) - \overline{\theta}\underline{D}(p(\overline{\theta})) \\
&\leq_{(iii)} \underline{V}(\underline{D}(\overline{\theta})) - \overline{\theta}\underline{D}(\overline{\theta}) \\
&= G^*.
\end{aligned}
$$

Inequality (i) follows from the fact that the right-hand-side is just expected welfare under a Dirac distribution that puts probability one at $\overline{\theta}$, when the demand is $\underline{D}$. Inequality (ii) follows from the fact that $\tilde{u}(\overline{\theta}, \underline{D}) \geq 0$ as $\widetilde{M}$ is EPIR. Inequality (iii) follows from the fact that

$$
\overline{\theta} = \arg\max_{\mathrm{p}} \left\{\underline{V}(\underline{D}(\mathrm{p})) - \overline{\theta}\underline{D}(\mathrm{p})\right\}.
$$

Next, to prove that $G(\widetilde{M}) = G^*$ for any $\widetilde{M} \in \widetilde{\mathcal{M}}^{\text{SL}}$, it suffices to note that there exists a mechanism $\underline{\widetilde{M}} \in \widetilde{\mathcal{M}}$ such that $G(\underline{\widetilde{M}}) = G^*$. Let $\underline{\widetilde{M}} \equiv (p, t)$ be the price mechanism such that $p(\theta) = \overline{\theta}$ for all $\theta \in \Theta$, and where $t(\theta, D)$ satisfies Condition (9) with $\tilde{u}(\overline{\theta}, D) = 0$ for all $\theta \in \Theta$, all $D \in \mathcal{D}$. Under such a mechanism, for all $\theta \in \Theta$, all $D \in \mathcal{D}$, welfare is equal to

$$V(D(\overline{\theta})) - \theta D(\overline{\theta}) - \int_{\theta}^{\overline{\theta}} D(\overline{\theta}) dy = V(D(\overline{\theta})) - \overline{\theta} D(\overline{\theta}) \geq \underline{V}(\underline{D}(\overline{\theta})) - \overline{\theta}\underline{D}(\overline{\theta}) = G^*.$$

In turn, this means that, for all $F \in \mathcal{F}$ and $D \in \mathcal{D}$, $\widetilde{W}(\underline{\widetilde{M}}; D, F) \geq G^*$, which implies that $G(\underline{\widetilde{M}}) = G^*$.

We now prove that, if $\widetilde{M} \equiv (p, \tilde{u}) \in \widetilde{\mathcal{M}}^{\text{SL}}$, Conditions (a)-(d) in the lemma must hold. By standard arguments, that $\widetilde{M}$ is EPIC and EPIR implies that p is weakly increasing and $\tilde{u}$ satisfies Condition (9) for all $\theta \in \Theta$ and $D \in \mathcal{D}$, with $\tilde{u}(\overline{\theta}, D) \geq 0$. That $p(\overline{\theta}) = \overline{\theta}$ and $\tilde{u}(\overline{\theta}, \underline{D}) = 0$ follows from the fact that the only way welfare can be equal to G^* when Nature selects $D = \underline{D}$ and a technology that selects $\theta = \overline{\theta}$ with probability one is by inducing efficient output by setting a price $p(\overline{\theta}) = \overline{\theta}$, and giving no rent to the monopolist, which amounts to setting $\tilde{u}(\overline{\theta}, \underline{D}) = 0$. That $\tilde{w}(\theta, \widetilde{M}; D) \geq G^*$ must hold for all $\theta \in \Theta$ all $D \in \mathcal{D}$ follows from the fact that, if this is not the case, then $G(\widetilde{M}) < G^*$, a contradiction to $\widetilde{M} \in \widetilde{\mathcal{M}}^{\text{SL}}$.

Finally, to see that, jointly, Conditions (a)-(d) imply that $\widetilde{M} \in \widetilde{\mathcal{M}}^{\text{SL}}$, observe that Conditions (a)-(c) imply that $\widetilde{M}$ is EPIC and EPIR. Furthermore, Condition (d) implies that $G(\widetilde{M}) \geq G^*$. By inequality (iii) above, we thus have that $G(\widetilde{M}) = G^*$. The first part of the lemma then implies that $\widetilde{M} \in \widetilde{\mathcal{M}}^{\text{SL}}$. ■

Proof of Proposition 3. By standard arguments, for any EPIC and EPIR price regulation $\widetilde{M} \equiv (p, t)$, expected welfare under the conjectured model $(D^\star, F^\star)$ is equal to

$$\int_{\underline{\theta}}^{\overline{\theta}} \Big[V^\star(D^\star(p(\theta))) - z^\star(\theta) D^\star(p(\theta)) \Big] F^\star(\mathrm{d}\theta) - \widetilde{u}(\overline{\theta}, D^\star).$$

Any robustly-optimal regulation thus maximizes this expression subject to the short-list constraints in Lemma 3.

Consider a relaxed program, where:

- the constraints in Conditions (a) and (c) are relaxed to $p(\theta) \leq \overline{\theta}$ for all $\theta \in \Theta$,

- the constraints on $\{\tilde{u}(\theta, D)\}_{D \in \mathcal{D}}$ in Condition (b) are replaced by $\widetilde{u}(\overline{\theta}, D^\star) \geq 0$,
- the constraints in Condition (d) are dropped.

Any solution to this relaxed program is such that $\widetilde{u}(\overline{\theta}, D^\star) = 0$ and, for any $\theta > \underline{\theta}$, $p(\theta)$ is as in the Baron-Myerson-with-price-cap regulation. This is because, for any $\theta \in \Theta$, the expression $V^\star(D^\star(\cdot)) - z^\star(\theta)D^\star(\cdot)$ is quasi-concave in p. The unique maximizer of this expression over $[0, \overline{\theta}]$ is thus $p(\theta) = \min\{z^\star(\theta), \overline{\theta}\}$. Because p is non-decreasing, any solution to the relaxed program is then such that $p(\theta) = \min\{z^\star(\theta), \overline{\theta}\}$ for all $\theta > \underline{\theta}$.

Equipped with this result, we now show that when the rent function $\tilde{u}$ satisfies the conditions in Definition 4, all the remaining properties of Lemma 3 are satisfied, implying that the mechanism is in the short list. First note that, Condition (b) in Lemma 3 trivially holds. Since $z^\star$ is increasing, Condition (a) is also satisfied. Since $z^\star(\overline{\theta}) > \overline{\theta}$, Condition (c) also holds. To complete the proof, it thus suffices to show that Constraint (d) is satisfied. Notice that Condition (11) implies that, when $\widetilde{M}$ is a Baron-Myerson-with-price-cap regulation $\widetilde{w}(\overline{\theta}, \widetilde{M}, D) \geq G^*$. Now, pick any $\theta < \overline{\theta}$. Observe that $z^\star(\theta) \geq \theta$, and hence, $D(z^\star(\theta)) \leq D(\theta)$ (i.e., for any demand curve D, the quantity traded under any Baron-Myerson-with-price-cap regulation when the cost is θ is below the efficient level $D(\theta)$). Lemma 8 then implies $\widetilde{w}(\cdot, \widetilde{M}; D)$ is weakly decreasing in θ. Thus, $\widetilde{w}(\theta, \widetilde{M}^{\text{OPT}}; D) \geq \widetilde{w}(\overline{\theta}, M; D) \geq G^*$. ■

Proof of Proposition 4. The proof is in two parts, each establishing the corresponding claim in the proposition.

Part (1). If $M^{\text{OPT}} = M^\star$, then $q^{\text{OPT}}(\theta) = \max\{q^{\text{BM}}(\theta), \mathrm{q}_\ell\}$ for all θ, where $\mathrm{q}_\ell \equiv \underline{D}(\overline{\theta})$ is the efficient quantity for cost $\overline{\theta}$ and demand $\underline{D}$. In this case, there exists $\theta^\star \leq \overline{\theta}$ such that $q^{\text{OPT}}(\theta) = \mathrm{q}_\ell$ if $\theta \geq \theta^\star$, and $q^{\text{OPT}}(\theta) = q^{\text{BM}}(\theta)$ if $\theta < \theta^\star$. See Figure 1 for the illustration.

Under any robustly optimal price regulation $\widetilde{M}^{\text{OPT}} = (p^{\text{OPT}}, \widetilde{u}^{\text{OPT}})$, when the demand is $D^\star$ and cost is θ, the monopolist sells a quantity $D^\star(p^{\text{OPT}}(\theta)) = \max\{q^{\text{BM}}(\theta), \tilde{\mathrm{q}}_\ell\}$, where $\tilde{\mathrm{q}}_\ell \equiv D^\star(\overline{\theta}) \geq \underline{D}(\overline{\theta}) = \mathrm{q}_\ell$. Thus, there exists $\tilde{\theta}^\star \leq \theta^\star$ such that $D^\star(p^{\text{OPT}}(\theta)) = \tilde{\mathrm{q}}_\ell$ if $\theta \geq \tilde{\theta}^\star$ and $D^\star(p^{\text{OPT}}(\theta)) = q^{\text{BM}}(\theta)$ if $\theta < \tilde{\theta}^\star$. See Panel A of Figure 4 for the illustration.

For $\theta < \tilde{\theta}^\star$, we have that $D^\star(p^{\text{OPT}}(\theta)) = q^{\text{OPT}}(\theta) = q^{\text{BM}}(\theta)$. However, for $\theta \geq \tilde{\theta}^\star$, we have that $D^\star(p^{\text{OPT}}(\theta)) = \tilde{\mathrm{q}}_\ell \geq q^{\text{OPT}}(\theta) \geq q^{\text{BM}}(\theta)$. Because, for any θ, virtual surplus $V^\star(\mathrm{q}) - z^\star(\theta)\mathrm{q}$ is quasi-concave in q, reaching a maximum at $q^{\text{BM}}(\theta)$, we thus have that, for any $\theta \geq \tilde{\theta}^\star$,

$$V^\star(D^\star(p^{\text{OPT}}(\theta))) - z^\star(\theta)D^\star(p^{\text{OPT}}(\theta)) \leq V^\star(q^{\text{OPT}}(\theta)) - z^\star(\theta)q^{\text{OPT}}(\theta).$$

We conclude that $\widetilde{W}(\widetilde{M}^{\text{OPT}}; D^\star, F^\star) \leq W(M^{\text{OPT}}; V^\star, F^\star)$, with the inequality strict if, and only if, $D^\star(\bar{\theta}) > \underline{D}(\bar{\theta})$.

Part (2). If $D^\star(\bar{\theta}) = \underline{D}(\bar{\theta})$, then $\mathrm{q}_\ell = \tilde{\mathrm{q}}_\ell$. In this case, for any $\theta \in \Theta$, the quantity traded under the conjectured model $(D^\star, F^\star)$ when running any robustly optimal price regulation is $D^\star(p^{\text{OPT}}(\theta)) = \max\{q^{\text{BM}}(\theta), \mathrm{q}_\ell\} = q^\star(\theta)$. This means that, by running any robustly optimal price regulation $\widetilde{M}^{\text{OPT}}$, the regulator obtains the same welfare as by running the Baron-Myerson-with-quantity-floor regulation $M^\star$, i.e.,

$$\widetilde{W}(\widetilde{M}^{\text{OPT}}; D^\star, F^\star) = W(M^\star; V^\star, F^\star). \tag{28}$$

As shown in the proof of Proposition 1, $M^\star$ is the solution to a relaxation of the full program yielding the robustly optimal quantity regulation, implying that

$$W(M^\star; V^\star, F^\star) \geq W(M^{\text{OPT}}; V^\star, F^\star). \tag{29}$$

When $M^{\text{OPT}} \neq M^\star$, the inequality in (29) is strict. Jointly, (28) and (29) imply that, when $M^{\text{OPT}} \neq M^\star$ and $D^\star(\bar{\theta}) = \underline{D}(\bar{\theta})$, price regulation strictly dominates quantity regulation. ■

Proof of Corollary 6. Both claims follow from combining Propositions 1 and 4. For the first claim, note that the conditions in the corollary imply that $M^\star = (q^\star, u^\star)$ is robustly optimal and $D^\star(\bar{\theta}) > \underline{D}(\bar{\theta})$. Indeed, for any $\theta > \underline{\theta}$, Condition (14) in the corollary is equivalent to Condition (8) in Proposition 1. For $\theta = \underline{\theta}$, the assumption that $D^\star(\underline{\theta}) = \underline{D}(\underline{\theta})$ in the corollary implies that Condition (7) in Proposition 1 is equivalent to

$$\int_{\underline{\theta}}^{\bar{\theta}} q^\star(y)dy \leq \int_{\underline{\theta}}^{\bar{\theta}} \underline{D}(y)dy,$$

which is satisfied when Condition (14) in the corollary holds.

Next, consider the second claim. The conditions in the corollary imply that $D^\star(\bar{\theta}) = \underline{D}(\bar{\theta})$ and $M^\star = (q^\star, u^\star)$ is not robustly optimal. In fact, Conditions (15) and (16) in the corollary imply that either Condition (7) or Condition (8) in Proposition 1 is violated. ■

D Technical Lemmas

Lemma 8 *Suppose $M \equiv (q,u)$ is an IC mechanism and $I \subseteq \Theta$ is an interval. Let $\underline{W}(\cdot,q)$ be the function defined, for all $\theta \in \Theta$, by*

$$\underline{W}(\theta,q) \equiv \underline{V}(q(\theta)) - \theta q(\theta) - \int_{\theta}^{\bar{\theta}} q(y)dy. \tag{30}$$

The following are true:

A. *Suppose $0 < q(\theta) \le \underline{D}(\theta)$ for all $\theta \in I$. Then $\underline{W}(\cdot,q)$ is weakly decreasing over I. If, in addition, q is decreasing with $q(\theta) < \underline{D}(\theta)$ for all $\theta \in I$, then $\underline{W}(\cdot,q)$ is decreasing over I.*

B. *Suppose $q(\theta) > \underline{D}(\theta)$ for all $\theta \in I$. Then, $\underline{W}(\cdot,q)$ is weakly increasing over I. If, in addition, q is decreasing over I, then $\underline{W}(\cdot,q)$ is increasing over I.*

Proof of Lemma 8. Pick $\theta, \theta' \in I$, with $\theta' < \theta$. Note that

$$\underline{W}(\theta',q) - \underline{W}(\theta,q) = \int_{q(\theta)}^{q(\theta')} \underline{P}(y)dy - \theta' q(\theta') + \theta q(\theta) - \int_{\theta'}^{\theta} q(y)dy. \tag{31}$$

Proof of Part (A). We consider two cases.

Case 1: $\underline{P}(q(\theta')) \ge \theta > \theta'$. Note that the right-hand-side of (31) equals to

$$\int_{q(\theta)}^{q(\theta')} (\underline{P}(y) - \theta)dy + \left\{ (\theta - \theta')q(\theta') - \int_{\theta'}^{\theta} q(y)dy \right\}. \tag{32}$$

The first term in (32) is non-negative because, for all $y \in (q(\theta), q(\theta'))$, $\underline{P}(z) > \underline{P}(q(\theta')) \ge \theta$, which follows from $\underline{P}$ being decreasing. Furthermore, if $q(\theta') > q(\theta)$, then the first term in (32) is positive. Next, observe that, because q is weakly decreasing, the expression in curly brackets in (32) is non-negative. Thus, we conclude that $\underline{W}(\theta',q) \ge \underline{W}(\theta,q)$, i.e., $\underline{W}(\cdot,q)$ is weakly decreasing over I (decreasing when q is decreasing over I).

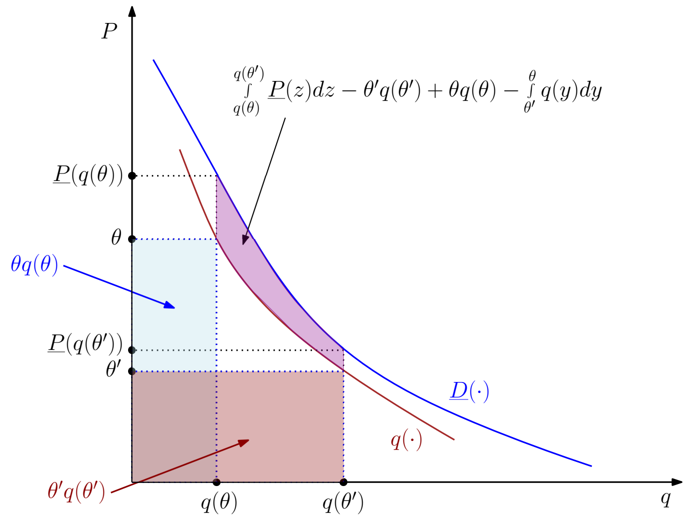


Figure 5: Illustration of Case 2 in Part A.

Case 2: $\theta > \underline{P}(q(\theta')) \geq \theta'$. Use Figure 5 to observe that the sum of the first four terms in (31) is equal to

$$\int_{\theta'}^{\underline{P}(q(\theta'))} \Big(q(\theta') - q(y)\Big)dy + \int_{\underline{P}(q(\theta'))}^{\theta} \Big(\underline{D}(y) - q(y)\Big)dy + \int_{\theta}^{\underline{P}(q(\theta))} \Big(\underline{D}(y) - q(\theta)\Big)dy. \tag{33}$$

Now we argue that each of these three terms in expression (33) is non-negative. The first term is non-negative because q is weakly decreasing. Next observe that, for all $y \in (\underline{P}(q(\theta')), \theta)$, $\underline{D}(y) \geq q(y)$. Hence, the second term in (33) is also non-negative. Finally, the last term in (33) is also non-negative because, for any $y \in (\theta, \underline{P}(q(\theta)))$, $\underline{D}(y) \geq q(\theta)$, which follows from $\underline{P}$ being decreasing. We conclude that $\underline{W}(\cdot, q)$ is weakly decreasing over I (decreasing when q is decreasing and such that $q(y) < \underline{D}(y)$ for all $y \in I$).

Proof of Part (B): The difference in welfare $\underline{W}(\theta, q) - \underline{W}(\theta', q)$ across the two types is given by the (negative of the) expression in (31), which can be rewritten as

$$\underline{W}(\theta, q) - \underline{W}(\theta', q) = \int_{\theta'}^{\theta} \Big(q(y) - q(\theta)\Big)dy + \int_{q(\theta)}^{q(\theta')} \Big(\theta' - \underline{P}(z)\Big)dz. \tag{34}$$

We consider two cases.

Case 1: $\underline{P}(q(\theta)) \leq \theta'$. In this case, $\underline{P}(z) < \theta'$ for all $z > q(\theta)$. This implies that the second integral in (34) is non-negative and the first integral is non-negative because q is weakly decreasing. If q is decreasing, both integrals are positive.

Case 2: $\theta' < \underline{P}(q(\theta)) < \theta$. We then have that $q(\theta') > \underline{D}(\theta') > q(\theta)$. Hence, using (34), we have that

$$\underline{W}(\theta, q) - \underline{W}(\theta', q) \geq \int_{\theta'}^{\theta} \Big(q(y) - q(\theta)\Big) dy - \int_{q(\theta)}^{\underline{D}(\theta')} \Big(\underline{P}(z) - \theta'\Big) dz. \tag{35}$$

See Figure 6 for an illustration of the right-hand-side of the inequality in (35). Changing the variable of integration, the second integral can be written as

$$\int_{q(\theta)}^{\underline{D}(\theta')} \Big(\underline{P}(z) - \theta'\Big) dz = \int_{\theta'}^{\underline{P}(q(\theta))} \Big(\underline{D}(y) - q(\theta)\Big) dy.$$

Thus, the right-hand-side of (35) reduces to (see Figure 6 for an illustration)

$$\int_{\theta'}^{\theta} \Big(q(y) - q(\theta)\Big) dy - \int_{\theta'}^{\underline{P}(q(\theta))} \Big(\underline{D}(y) - q(\theta)\Big) dy, \tag{36}$$

which is non-negative because $\underline{P}(q(\theta)) < \theta$ and $\underline{D}(y) < q(y)$ for all $y \in I$. Thus, $\underline{W}(\theta, q) \geq \underline{W}(\theta', q)$, i.e., $\underline{W}(\cdot, q)$ is weakly increasing over I. The above inequality also reveals that, when q is decreasing, the expression in (36) is positive, implying that $\underline{W}(\cdot, q)$ is increasing over I.

■

Proof of Lemma 4. (1) We want to establish that, for any $\theta \in \Theta$, constraints (4) and (5) are equivalent. Observe that, for any $\theta \in \Theta$,

$$\underline{V}(q(\theta)) - \theta q(\theta) = \int_{\theta}^{\infty} \underline{D}(y) dy - \underline{\mathrm{DWL}}(\theta, q(\theta)), \tag{37}$$

where, for any q, $\underline{\mathrm{DWL}}(\theta, \mathrm{q}) \equiv \int_{\theta}^{\underline{P}(\mathrm{q})} \big(\underline{D}(y) - \mathrm{q}\big) dy \geq 0$ is the dead-weight loss of procuring quantity q instead of quantity $\underline{D}(\theta)$ when the marginal cost is θ and the demand is $\underline{D}$. The equivalence between the constraints in (4) and (5) then follows from this observation together with the fact that

$$G^* \equiv \underline{V}(\mathrm{q}_\ell) - \overline{\theta}\mathrm{q}_\ell = \underline{V}(\underline{D}(\overline{\theta})) - \overline{\theta}\underline{D}(\overline{\theta}) = \int_{\overline{\theta}}^{\infty} \underline{D}(y) dy. \tag{38}$$

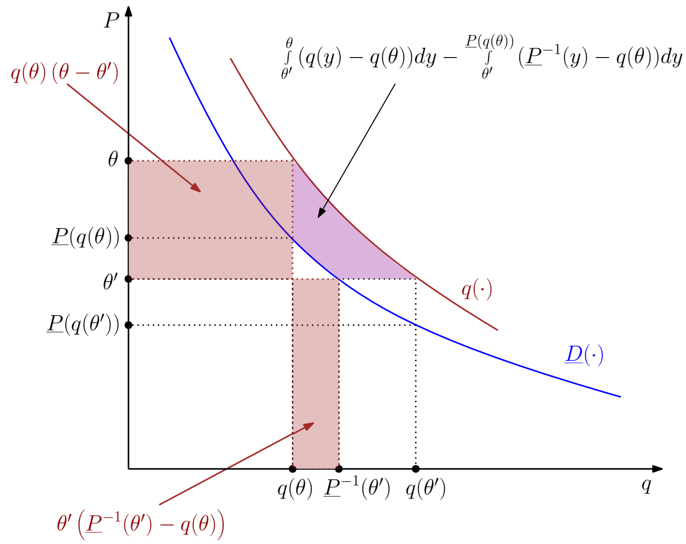


Figure 6: Illustration of Case 2 in Part B.

(2) Next, we establish that the following two statements are equivalent: (a) Condition (5) holds for all $\theta \in \Theta$; (b) Condition (5) holds for $\theta \in \{\underline{\theta}, \overline{\theta}\}$ and, in addition, Condition (20) holds for all $\theta \in (\underline{\theta}, \overline{\theta})$.

That (a) implies Condition (b) is immediate given that, for all θ, $\underline{\text{DWL}}(\theta, q(\theta)) \geq 0$. Thus, to complete the proof, it suffices to show that, when Condition (5) holds for $\theta \in \{\underline{\theta}, \overline{\theta}\}$ and, in addition, Condition (20) holds for all $\theta \in (\underline{\theta}, \overline{\theta})$, the following inequality

$$\int_{\theta}^{\overline{\theta}} \underline{D}(y)dy - \int_{\theta}^{\overline{\theta}} q(y)dy - \int_{\theta}^{\underline{P}(q(\theta))} \big(\underline{D}(y) - q(\theta)\big)dy \geq 0 \tag{39}$$

holds for all $\theta \in (\underline{\theta}, \overline{\theta})$. Below, we consider two cases, which are covered in Claims 1 and 2 below.

Claim 1 *Suppose $q(\theta) \leq \underline{D}(\theta)$ and, for all $\theta' \in \Theta$, $\int_{\theta'}^{\overline{\theta}} q(y)dy \leq \int_{\theta'}^{\overline{\theta}} \underline{D}(y)dy$. Then the inequality in (39) holds.*

Proof: Observe that $\theta \leq \underline{P}(q(\theta)) \leq \overline{\theta}$. The first inequality follows from the fact that $q(\theta) \leq \underline{D}(\theta)$. The second inequality follows from Condition (5) applied to $\theta = \overline{\theta}$ which gives $q(\overline{\theta}) = \mathrm{q}_\ell = \underline{D}(\overline{\theta})$; because q is weakly decreasing, we then have that $q(\theta) \geq \underline{D}(\overline{\theta})$. The left-hand-side of (39) is thus equivalent to

$$\int_{\underline{P}(q(\theta))}^{\overline{\theta}} (\underline{D}(y) - q(y))dy + \int_{\theta}^{\underline{P}(q(\theta))} (q(\theta) - q(y))dy. \tag{40}$$

That the first integral in (40) is positive follows from the fact that $\int_{\underline{P}(q(\theta))}^{\overline{\theta}} (\underline{D}(y) - q(y))dy \geq 0$. The second integral is non-negative because q is weakly decreasing. ■

Claim 2 *Suppose $q(\theta) > \underline{D}(\theta)$, and, for all $\theta' \in \Theta$, $\int_{\theta'}^{\overline{\theta}} q(y)dy \leq \int_{\theta'}^{\overline{\theta}} \underline{D}(y)dy$. Then the inequality in (39) holds.*

Proof: Because $q(\theta) > \underline{D}(\theta)$, $\underline{P}(q(\theta)) < \theta$, which implies that the left-hand-side of the inequality in (39) is equal to

$$\int_{\theta}^{\overline{\theta}} (\underline{D}(y) - q(y))\, dy - \int_{\underline{P}(q(\theta))}^{\theta} \big(q(\theta) - \underline{D}(y)\big)dy.$$

Let $\theta^\sharp \equiv \inf\{y \leq \theta : q(s) \geq \underline{D}(s) \, \text{for all} \, s \in (y, \theta]\}$ and note that $\theta^\sharp < \theta$. Suppose that $\theta^\sharp > \underline{\theta}$, which implies that $q(\theta^\sharp) = \underline{D}(\theta^\sharp)$. Because $\int_{\theta^\sharp}^{\overline{\theta}} (\underline{D}(y) - q(y))\, dy \geq 0$, we have that

$$\int_{\theta}^{\overline{\theta}} (\underline{D}(y) - q(y))\, dy \geq \int_{\theta^\sharp}^{\theta} (q(y) - \underline{D}(y))\, dy. \tag{41}$$

It follows that

$$\begin{aligned} &\int_{\theta}^{\overline{\theta}} (\underline{D}(y) - q(y))\, dy - \int_{\underline{P}(q(\theta))}^{\theta} \big(q(\theta) - \underline{D}(y)\big)dy \\ \geq &\int_{\theta^\sharp}^{\theta} (q(y) - \underline{D}(y))\, dy - \int_{\underline{P}(q(\theta))}^{\theta} \big(q(\theta) - \underline{D}(y)\big)dy. \end{aligned} \tag{42}$$

Because $q(\theta) \leq q(\theta^\sharp) = \underline{D}(\theta^\sharp)$, we have that $\underline{P}(q(\theta)) \geq \theta^\sharp$. This property, together with the fact that $q(y) > \underline{D}(y)$ for all $y \in (\theta^\sharp, \theta)$, implies that

$$\int_{\theta^\sharp}^{\theta} (q(y) - \underline{D}(y))\, dy \geq \int_{\underline{P}(q(\theta))}^{\theta} (q(y) - \underline{D}(y))\, dy.$$

In turn, this means that the right-hand-side of the inequality in (42) is greater than

$$\int_{\underline{P}(q(\theta))}^{\theta} (q(y) - \underline{D}(y))\, dy - \int_{\underline{P}(q(\theta))}^{\theta} \big(q(\theta) - \underline{D}(y)\big) dy = \int_{\underline{P}(q(\theta))}^{\theta} (q(y) - q(\theta))\, dy \geq 0,$$

where the last inequality follows from the fact that q is weakly decreasing. We thus conclude that the inequality in (39) holds, as claimed.

Next, suppose that $\theta^\sharp = \underline{\theta}$. That Condition (5) holds for $\theta = \underline{\theta}$ means that $\underline{W}(\underline{\theta}, q) \geq G^*$, where $\underline{W}(\cdot, q)$ is the function defined in (30). Because $q(y) \geq \underline{D}(y)$ for all $y \in [\underline{\theta}, \theta]$, Lemma 8 implies that the function $\underline{W}(\cdot, q)$ is weakly increasing over $[\underline{\theta}, \theta]$, which means that $\underline{W}(\theta, q) \geq G^*$. As shown above, this means that (39) holds. ■

Claims 1 and 2 cover the exhaustive cases and complete the proof of Lemma 4. ■

Lemma 9 *The following are true:*

$$\bar{z}^M(\theta) = z^\star(\theta) \qquad \forall\, \theta \in (\underline{\theta}, \theta_{\text{inf}}) \tag{L}$$

$$\bar{z}^M(\theta) > z^\star(\theta). \qquad \forall\, \theta \in (\theta_{\text{inf}}, \theta^\star) \tag{R}$$

Proof: Observe that z^M is right-continuous. This follows from the fact that $z^\star$ is continuous, which implies that $f^\star$ is also continuous, along with the fact that Λ^{OPT} is right-continuous.

We first establish (**L**) and then (**R**).

PROOF OF (**L**). If $\theta_{\text{inf}} = \underline{\theta}$, there is nothing to prove because $(\underline{\theta}, \theta_{\text{inf}})$ is an empty set. Thus assume that $\theta_{\text{inf}} > \underline{\theta}$. Suppose, for towards a contradiction, that there exists $\theta' \in (\underline{\theta}, \theta_{\text{inf}})$ such that $\bar{z}^M(\theta') \neq z^\star(\theta')$. Because $z^\star(\theta) = z^M(\theta)$ for all $\theta \in (\underline{\theta}, \theta_{\text{inf}})$, we have that $\bar{z}^M(\theta') \neq z^M(\theta')$, which implies $\bar{z}^M$ must be constant on a non-empty interval containing θ'. Let $K \equiv \bar{z}^M(\theta')$ and define

$$a := \inf\{\theta : \bar{z}^M(\theta) = K\}, \tag{43}$$

$$b := \sup\{\theta : \bar{z}^M(\theta) = K\}. \tag{44}$$

Hence, (a, b) is the largest interval where $\bar{z}^M$ takes value K. Because $\theta' < \theta_{\text{inf}}$ and $z^M(\theta) = z^\star(\theta)$ for all $\theta \in (\underline{\theta}, \theta_{\text{inf}})$ and $z^M(\theta) > z^\star(\theta)$ for all $\theta > \theta_{\text{inf}}$, there must exist $0 \leq \tilde{\epsilon} < b - a$ such that

$$z^M(\theta) = z^\star(\theta) \qquad \forall\, \theta \in (a, a + \tilde{\epsilon}), \tag{45}$$

$$z^M(\theta) \geq z^\star(\theta) \qquad \forall\ \theta \in (a+\tilde{\epsilon}, b). \tag{46}$$

Below, we use the following claim to establish the result.

Claim 3 *There exist two sequences* $\{\epsilon_n\} \downarrow 0$ *and* $\{\delta_n\} \downarrow 0$*, and* $\overline{N} \in \mathbb{N}$ *such that*

$$z^M(a+\epsilon_n) > K > z^M(b-\delta_n) \qquad \forall\ n \geq \overline{N}.$$

Proof of Claim 3: By the definition of the H and $\overline{H}$ functions, we must have that $H(F^\star(a)) = \overline{H}(F^\star(a))$ and $H(F^\star(b)) = \overline{H}(F^\star(b))$. Now assume, for contradiction, that there does not exist a sequence $\{\epsilon_n\} \downarrow 0$ such that $z^M(a+\epsilon_n) > K$. This means that there exists $0 < \bar{\epsilon} < b-a$ such that[16]

$$z^M(a+\epsilon) \leq K \qquad \forall \epsilon \in [0, \bar{\epsilon}),$$

or, equivalently, $h(\phi) \leq \bar{h}(\phi)$ for all $\phi \in [F^\star(a), F^\star(a+\bar{\epsilon}))$. This inequality, combined with $H(F^\star(a)) = \overline{H}(F^\star(a))$, implies that, for some $\epsilon \in (0, \bar{\epsilon})$,

$$H(F^\star(a)) + \int_{F^\star(a)}^{F^\star(a+\epsilon)} h(\phi)\,\mathrm{d}\phi \leq \overline{H}(F^\star(a)) + \int_{F^\star(a)}^{F^\star(a+\epsilon)} \bar{h}(\phi)\,\mathrm{d}\phi$$
$$\Longrightarrow H(F^\star(a+\epsilon)) \leq \overline{H}(F^\star(a+\epsilon)).$$

The last inequality contradicts the fact that $H(F^\star(\theta)) > \overline{H}(F^\star(\theta))$ for all $\theta \in (a, b)$.

Similarly assume, for contradiction, that there does not exist a sequence $\{\delta_n\} \downarrow 0$ such that $z^M(b-\delta_n) < K$. This means that there exists $0 < \bar{\epsilon} < b-a$ such that

$$K \leq z^M(b-\epsilon) \qquad \forall \epsilon \in (0, \bar{\epsilon}),$$

or, equivalently, $\bar{h}(\phi) \leq h(\phi)$ for $\phi \in (F^\star(b-\bar{\epsilon}), F^\star(b))$. Observe that $H(F^\star(b)) = \overline{H}(F^\star(b))$. Therefore, for some $\epsilon \in (0, \bar{\epsilon})$,

$$H(F^\star(b-\epsilon)) + \int_{F^\star(b-\epsilon)}^{F^\star(b)} h(\phi)\,d\phi = \overline{H}(F^\star(b-\epsilon)) + \int_{F^\star(b-\epsilon)}^{F^\star(b)} \bar{h}(\phi)\,d\phi.$$

[16] Otherwise, for every $\delta > 0$ there exists $\epsilon \in (0, \delta)$ such that $z^M(a+\epsilon) > K$. We can use this to create a sequence $\{\epsilon_n\} \downarrow 0$ that satisfies the properties in the claim.

Because $\bar{h}(\phi) \leq h(\phi)$ for all $\phi \in (F^\star(b-\bar{\epsilon}), F^\star(b))$, we have that

$$\int_{F^\star(b-\epsilon)}^{F^\star(b)} h(\phi)\, d\phi \geq \int_{F^\star(b-\epsilon)}^{F^\star(b)} \bar{h}(\phi)\, d\phi.$$

Therefore $H(F^\star(b-\epsilon)) \leq \overline{H}(F^\star(b-\epsilon))$, a contradiction to the assumption that $H(F^\star(\theta)) > \overline{H}(F^\star(\theta))$ for all $\theta \in (a,b)$. This completes the proof of Claim 3. ■

Claim 3 implies that there exists $\epsilon, \delta > 0$ small such that

$$z^M(a+\epsilon) > K > z^M(b-\delta).$$

Note that ϵ and δ can be chosen so that $b-\delta > a+\epsilon$. By (45) and (46), we have that $z^\star(a+\epsilon) > K > z^\star(b-\delta)$, which contradicts the fact that $z^\star$ is increasing. We thus conclude that $\bar{z}^M(\theta) = z^\star(\theta)$ for all $\theta \in (\underline{\theta}, \theta_{\text{inf}})$.

Proof of (**R**). Next, we establish that $\bar{z}^M(\theta) > z^\star(\theta)$ for $\theta \in (\theta_{\text{inf}}, \theta^\star)$. Suppose there exists $\theta' \in (\theta_{\text{inf}}, \theta^\star)$ such that $\bar{z}^M(\theta') \leq z^\star(\theta')$. Because $\theta' > \theta_{\text{inf}}$, we have that $z^\star(\theta') < z^M(\theta')$. This further implies that $\bar{z}^M(\theta') < z^M(\theta')$. This can only happen if $\bar{z}^M$ is locally constant over a non-empty interval containing θ'. Let $\bar{z}^M(\theta') \equiv K$, and define the open neighborhood (a,b) as in (43) and (44). We consider two cases.

Case 1. $\theta' < b$. By the definition of the H and $\overline{H}$ functions, we must have that $H(F^\star(b)) = \overline{H}(F^\star(b))$. Because $b > \theta' > \theta_{\text{inf}}$, Claim 3 above implies there exists $\epsilon > 0$ small such that $\theta' < b-\epsilon$ and $\bar{z}^M(\theta') = K > z^M(b-\epsilon)$. By assumption $z^\star(\theta') \geq \bar{z}^M(\theta')$. Furthermore, $z^M(b-\epsilon) > z^\star(b-\epsilon)$ because $z^M(\theta) > z^\star(\theta)$ for all $\theta > \theta_{\text{inf}}$. This implies that

$$z^\star(\theta') \geq \bar{z}^M(\theta') > z^M(b-\epsilon) > z^\star(b-\epsilon),$$

which contradicts the assumption that $z^\star$ is increasing.

Case 2: Suppose that $b = \theta' > \theta_{\text{inf}}$. Right-continuity of z^M and $\bar{z}^M$, along with the definition of b, imply that

$$\bar{z}^M(\theta') = \lim_{y \downarrow \theta'} \bar{z}^M(y) = \lim_{y \downarrow \theta'} z^M(y) = z^M(\theta').$$

Because $z^M(\theta') > z^\star(\theta')$, this contradicts the assumption that $\bar{z}^M(\theta') \leq z^\star(\theta')$.

We thus conclude that, for all $\theta' \in (\theta_{\inf}, \theta^\star)$, $\bar{z}^M(\theta) > z^\star(\theta)$, as claimed. ■

Lemma 10 *If $\theta^m < \overline{\theta}$, then $q^\star(\theta^m) \geq \underline{D}(\theta^m)$. In addition, if $\theta^m > \underline{\theta}$, then $q^\star(\theta^m) = \underline{D}(\theta^m)$.*

Proof: By Lemma 5, if $\theta^m < \overline{\theta}$, then $\theta^m < \theta^\star$, and therefore $q^\star(\theta^m) = q^{\mathrm{BM}}(\theta^m)$. Suppose that $q^\star(\theta^m) < \underline{D}(\theta^m)$. Because $\underline{D}$ is continuous and $q^\star$ is weakly decreasing and continuous, and $\theta^m < \theta^\star$, there exists $\delta > 0$ such that, for all $\theta \in [\theta^m, \theta^m + \delta]$, $0 < \mathrm{q}_\ell < q^\star(\theta) < \underline{D}(\theta)$. Also, for small enough δ, we have $\theta^m + \delta < \theta^\star$. This means that $q^\star(\theta) = q^{\mathrm{BM}}(\theta)$ for all $\theta \in [\theta^m, \theta^m + \delta]$. By regularity, q^{BM} is decreasing, implying that $q^\star$ is decreasing in this interval. Part A of Lemma 8 then implies that the function $\underline{W}(\cdot, q^\star)$ is decreasing over $[\theta^m, \theta^m + \delta]$, contradicting the definition of θ^m. Hence, $q^\star(\theta^m) \geq \underline{D}(\theta^m)$.

Similarly, if $\theta^m > \underline{\theta}$ and $q^\star(\theta^m) > \underline{D}(\theta^m)$, there exist $\delta > 0$ and a left-neighborhood $[\theta^m - \delta, \theta^m]$, such that for every θ in this neighborhood $q^\star(\theta) > \underline{D}(\theta)$. Also $q^\star(\theta) = q^{\mathrm{BM}}(\theta)$ in $[\theta^m - \delta, \theta^m]$ (since $\theta^m < \theta^\star$), and q^{BM} is decreasing by regularity. Hence, by Part B of Lemma 8, $\underline{W}(\cdot, q^\star)$ is increasing in $[\theta^m - \delta, \theta^m]$, contradicting the definition of θ^m. ■

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